Sample Solutions for Sample Questions # 6

1.

$$
\begin{array}{c}
\qquad A \qquad\qquad\qquad I_4 \\
\left(\begin{array}{cccc|cccc}
1 & 1 & -3 & 1 & 1 & & & \\
1 & 2 & -4 & 2 & & 1 & & \\
2 & 3 & -6 & 5 & & & 1 & \\
3 & 3 & -9 & 4 & & & & 1
\end{array}\right)
\end{array}
\longrightarrow
\left(\begin{array}{cccc|cccc}
1 & 1 & -3 & 1 & 1 & & & \\
0 & 1 & -1 & 1 & -1 & 1 & & \\
0 & 1 & 0 & 3 & -2 & & 1 & \\
0 & 0 & 0 & 1 & -3 & & & 1
\end{array}\right)
$$

$$
\longrightarrow
\left(\begin{array}{cccc|cccc}
1 & 1 & -3 & 1 & 1 & & & \\
0 & 1 & -1 & 1 & -1 & 1 & & \\
0 & 0 & 1 & 2 & -1 & -1 & 1 & \\
0 & 0 & 0 & 1 & -3 & & & 1
\end{array}\right)
\longrightarrow
\left(\begin{array}{cccc|cccc}
1 & 1 & -3 & 0 & 4 & & & -1 \\
0 & 1 & -1 & 0 & 2 & 1 & & -1 \\
0 & 0 & 1 & 0 & 5 & -1 & 1 & -2 \\
0 & 0 & 0 & 1 & -3 & & & 1
\end{array}\right)
$$

$$
\longrightarrow
\left(\begin{array}{cccc|cccc}
1 & 1 & 0 & 0 & 19 & -3 & 3 & -7 \\
0 & 1 & 0 & 0 & 7 & 0 & 1 & -3 \\
0 & 0 & 1 & 0 & 5 & -1 & 1 & -2 \\
0 & 0 & 0 & 1 & -3 & & & 1
\end{array}\right)
\longrightarrow
\begin{array}{c}
\left(\begin{array}{cccc|cccc}
1 & & & & 12 & -3 & 2 & -4 \\
 & 1 & & & 7 & 0 & 1 & -3 \\
 & & 1 & & 5 & -1 & 1 & -2 \\
 & & & 1 & -3 & 0 & 0 & 1
\end{array}\right) \\
\qquad\qquad I_4 \qquad\qquad\qquad B.
\end{array}
$$

Check:

$$
\begin{pmatrix}
12 & -3 & 2 & -4 \\
7 & 0 & 1 & -3 \\
5 & -1 & 1 & -2 \\
-3 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -3 & 1 \\
1 & 2 & -4 & 2 \\
2 & 3 & -6 & 5 \\
3 & 3 & -9 & 4
\end{pmatrix}
= I_4 .
$$

$$
\begin{pmatrix}
1 & 1 & -3 & 1 \\
1 & 2 & -4 & 2 \\
2 & 3 & -6 & 5 \\
3 & 3 & -9 & 4
\end{pmatrix}
\begin{pmatrix}
12 & -3 & 2 & -4 \\
7 & 0 & 1 & -3 \\
5 & -1 & 1 & -2 \\
-3 & 0 & 0 & 1
\end{pmatrix}
= I_4 .
$$

2. (a)

$$0 = 0 + 0x + 0x^2 + 0x^3.$$

verify:

$$(0 + 0x + 0x^2 + 0x^3) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3)$$

$$= (0 + a_0) + (0 + a_1)x + (0 + a_2)x^2 + (0 + a_3)x^3$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

(b) zero matrix $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

verify:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

$$= \begin{pmatrix} 0 + a_{11} & 0 + a_{12} & 0 + a_{13} & 0 + a_{14} \\ 0 + a_{21} & 0 + a_{22} & 0 + a_{23} & 0 + a_{24} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

(c) zero $= f : [0,1] \to \mathbb{R}$.
$x \longmapsto 0$.

verify:

$(f+g)(x) = f(x) + g(x)$ $\Rightarrow f + g = g$

$= g(x)$ for all $x \in [0,1]$.

(d) zero $= f : \mathbb{N} \to \mathbb{R}$.
$x \longmapsto 0$.

verify:

$(f+g)(x) = f(x) + g(x)$

$= g(x)$ for all $x \in \mathbb{N}$. $\Rightarrow f + g = g$

3. (a) $v = -3 - 2x + x^2$

additive inverse $w = 3 + 2x - x^2$.

verify:

$(-3 - 2x + x^2) + (3 + 2x - x^2)$

$= (3+3) + (-2+2)x + (1-1)x^2$

$= 0 + 0x + 0x^2 + 0x^3$ ← zero in the space.

(b). $v = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$

additive inverse $w = \cancel{\begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}}$. $\begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$

verify:

$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} (1-1) & (-1+1) \\ (0+0) & (3-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

↑ zero in the space.

(c). $v = 3e^x - 2e^{-x}$

additive inverse $w = -3e^x + 2e^{-x}$.

verify:

$3e^x - 2e^{-x} - 3e^x + 2e^{-x} = 0$ ← zero in the space.

4. Goal: ① if $\vec{b}=0$, then V is a vector space.

② if $\vec{b}\neq 0$, then V is not a vector space.

①. Verify all 10 properties:

Suppose $\vec{x}, \vec{y}, \vec{z} \in V$ and $r, s \in \mathbb{R}$.

(1) $A\vec{x}=\vec{0}$, $A\vec{y}=\vec{0}$

$\Rightarrow A(\vec{x}+\vec{y})=\vec{0}$, so $\vec{x}+\vec{y} \in V$.

(2) $\vec{x}+\vec{y}=\vec{y}+\vec{x}$ in $\mathbb{R}^n$.

(3) $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$ in $\mathbb{R}^n$.

(4) $\vec{0}\in V$ because $A\vec{0}=\vec{0}$.

also, $\vec{0}+\vec{x}=\vec{x}$ in $\mathbb{R}^n$.

(5) $-\vec{x}\in V$ because $A(-\vec{x})=-\vec{0}=\vec{0}$.

and $-\vec{x}+\vec{x}=\vec{0}$.

(6) $A\vec{x}=\vec{0} \Rightarrow A(r\vec{x})=r\cdot A\vec{x}=\vec{0}$

, so $r\vec{x}\in V$.

(7) $(r+s)\cdot\vec{x}=r\vec{x}+s\vec{x}$ in $\mathbb{R}^n$.

(8) $r\cdot(\vec{x}+\vec{y})=r\vec{x}+r\vec{y}$ in $\mathbb{R}^n$

(9) $(r\cdot s)\cdot\vec{x}=r\cdot(s\cdot\vec{x})$ in $\mathbb{R}^n$.

(10) $1\cdot\vec{x}=\vec{x}$ in $\mathbb{R}^n$.

②. If $\vec{b}\neq\vec{0}$, then $\vec{b}+\vec{b}\neq\vec{b}$.

Suppose $\vec{x}, \vec{y}\in V$. Then $A\vec{x}=\vec{b}$ and $A\vec{y}=\vec{b}$.

However

$A(\vec{x}+\vec{y})=A\vec{x}+A\vec{y}=\vec{b}+\vec{b}\neq\vec{b}$.

$\Rightarrow \vec{x}+\vec{y}\notin V$. and thus (1) fails.

5. Verify all 10 properties:

Suppose $X_1, X_2, X_3 \in V$ and $r, s \in \mathbb{R}$.

(1) $AX_1 = 0$, $AX_2 = 0$

$\Rightarrow A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$.

so $X_1 + X_2 \in V$.

(2) $X_1 + X_2 = X_2 + X_1$ in $M_{n\times n}$

(3) $(X_1 + X_2) + X_3 = X_1 + (X_2 + X_3)$ in $M_{n\times n}$

(4) $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{n\times n}$

and $A\,0 = 0$, so $0 \in V$.

(5) $-X_1 \in V$ because $A(-X_1) = -AX_1 = -0 = 0$.

(6) $AX_1 = 0 \Rightarrow A(rX_1) = rAX_1 = 0$.

so $rX_1 \in V$.

(7) $(r+s)\cdot X_1 = rX_1 + sX_1$ in $M_{n\times n}$

(8) $r(X_1 + X_2) = rX_1 + rX_2$ in $M_{n\times n}$

(9) $(r\cdot s)\cdot X_1 = r\cdot(s\cdot X_1)$ in $M_{n\times n}$

(10) $1\cdot X_1 = X_1$ in $M_{n\times n}$.

6. (a). $\cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}$ ~~in~~ +

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ are in the space because $1+0+0=1$, $0+1+0=\cancel{0}\,1$

But $\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}$ is not in the space because $1+1+0=2\neq 1$.

$\Rightarrow$ (1) fails.

(b) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ are in the space because $1^2+0^2+0^2=1$, $0^2+1^2+0^2=1$.

But $\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}$ is not in the space because $1^2+1^2+0^2=2\neq 1$.

$\Rightarrow$ (1) fails.

(c). Pick $\underset{\text{vector}}{5}\in\mathbb{R}^+$ and a scalar $-1\in\mathbb{R}$.

Then $(-1)\cdot 5=-5\notin\mathbb{R}^+$.

$\Rightarrow$ (6) fails.

7. Claim: $(\mathbb{R}^+, \oplus, \otimes)$ is a vector space

when $x \oplus y = x \cdot y$ and $r \otimes x = x^r$.

Verify all 10 properties:

Suppose $x, y, z \in \mathbb{R}^+$, $r, s \in \mathbb{R}$.

(1) $x, y > 0$

then $x \oplus y = x \cdot y > 0$

$\Rightarrow x \oplus y \in \mathbb{R}^+$.

(2) $x \oplus y = x \cdot y = y \cdot x = y \oplus x$.

(3) $(x \oplus y) \oplus z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \oplus (y \oplus z)$.

(4) 1 has the property that

$1 \oplus x = 1 \cdot x = x$

$\Rightarrow$ 1 is the zero vector in $\mathbb{R}^+$.

(5). If $x \in \mathbb{R}^+$, then $\frac{1}{x} \in \mathbb{R}^+$.

Also $\frac{1}{x} \oplus x = \frac{1}{x} \cdot x = 1 \leftarrow$ zero in $\mathbb{R}^+$.

so $\frac{1}{x}$ is the additive inverse of x under $\oplus$.

(6) $x > 0 \Rightarrow x^r > 0$

then $r \otimes x = x^r \in \mathbb{R}^+$.

(7) $(r+s) \otimes x = x^{r+s} = x^r \cdot x^s = (r \otimes x) \oplus (s \otimes x)$

addition in $\mathbb{R}$ — addition in $\mathbb{R}^+$

(8) $r \otimes (x \oplus y) = (x \cdot y)^r = x^r \cdot y^r = \cancel{r \otimes x}$ $(r \otimes x) \oplus (r \otimes y)$.

(9) $(r \cdot s) \otimes x = x^{rs} = (x^s)^r = r \otimes (s \otimes x)$

multiplication in $\mathbb{R}$ — scalar multiplication

(10) The scalar $1 \in \mathbb{R}$ has the property

$1 \otimes x = x^1 = x$.