Sample Solutions for Sample Questions ~~5~~. 7

1.

(a). Yes. $S = \text{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right)$.

[Since S is the span of some vectors, S is a vector space.]

(b) No. S does not contain a zero vector.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S.$$

(c). Yes. $S = \text{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\right)$.

To solve $a+b=0$,

let $b = s$ $\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -s \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

2. (a). Intuitively, $v = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \text{span}(S)$.

In general, solve

$$a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow a = 2, \ b = 1.$$

(b). Solve

$$
\begin{array}{rl}
 & a\cdot(0+0x+1\cdot x^2+0x^3) \\
 & b\cdot(0+2x+1x^2+0x^3) \\
+) & c\cdot(0+1x+0x^2+1\cdot x^3) \\
\hline
 & 0+1\cdot x+0.x^2-1\cdot x^3.
\end{array}
$$

$$
\Longleftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \begin{array}{l} a=-1 \\ b=1 \\ c=-1 \end{array}
$$

So

$$
\begin{aligned}
x-x^3 = \overset{-1}{\cancel{a}}\cdot(&\quad x^2 \quad) \\
+1\cdot(&\quad 2x+x^2 \quad) \\
-1\cdot(&\quad x \quad + x^3) \quad \in \text{span}(S).
\end{aligned}
$$

(c) Solve $a\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}.$

$$
\Longleftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \text{no solution.}
$$

So $v \notin \text{span}(S)$.

3.

(a) Solve
$$a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\2\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\3\end{pmatrix}=\underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix}}_{\text{any vector in } \mathbb{R}^3.}$$

$$\Longleftrightarrow \quad \begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

augmented matrix

$$\left(\begin{array}{ccc|c}1&&&x\\&2&&y\\&&3&z\end{array}\right)$$

always has solution regardless of the choice of $\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Yes, $\text{span}(S) = \mathbb{R}^3$.

(b). Solve
$$\begin{pmatrix}1&0&2\\0&1&3\\1&1&5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

augmented matrix

$$\left(\begin{array}{ccc|c}1&0&2&x\\0&1&3&y\\1&1&5&z\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&2&x\\0&1&3&y\\0&1&3&z-x\end{array}\right)$$

$$\longrightarrow\left(\begin{array}{ccc|c}1&0&2&x\\0&1&3&y\\0&0&0&z-x-y\end{array}\right)$$

The linear system has no solution if $z-x-y \neq 0$,

so $\text{span}(S) \neq \mathbb{R}^3$.

[e.g. $\begin{pmatrix}1\\0\\0\end{pmatrix} \notin \text{span}(S)$ since $z-x-y = 0-1-0=-1 \neq 0$]

(c)

Solve $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & x-y \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & x-y \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z - \frac{1}{2}(x-y) \end{array}\right)$$ always has

Solution regardless of the choice of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

So $\text{span}(S) = \mathbb{R}^3$.

4. Let $E = \{\text{all even functions}\}$.

$\mathcal{O} = \{\text{all odd functions}\}$.

① Claim: E is a subspace.

Suppose f, g are even functions and $r \in \mathbb{R}$.

Then $\begin{cases} f(-x) = f(x) \\ g(-x) = g(x) \end{cases}$

So $f(-x) + g(-x) = f(x) + g(x) \Leftrightarrow f+g \in E$.

$r \cdot f(-x) = f(x) \Leftrightarrow rf \in E$.

② Claim: $\mathcal{O}$ is a subspace.

Suppose $f, g \in \mathcal{O}$ and $r \in \mathbb{R} \Rightarrow \begin{cases} f(-x) = -f(x) \\ g(-x) = -g(x). \end{cases}$

$\Rightarrow$ $f(-x) + g(-x) = -[f(x) + g(x)] \Rightarrow f+g \in \mathcal{O}$.

$rf(-x) = -rf(x) \Rightarrow rf \in \mathcal{O}$.

5. Equivalently, we show

$c \cdot v = 0$ for all $v \in S \iff c \cdot v = 0$ for all $v \in \text{span}(S)$.

"$\Rightarrow$" Let $u \in \text{span}(S)$.

Then $u = a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k v_k$, $a_1, \dots, a_k \in \mathbb{R}$, $v_1, \dots, v_k \in S$.

We know $c \cdot v_i = 0$ for all v_i.

$$\Rightarrow c \cdot u = c \cdot (a_1 v_1 + a_2 \cdot v_2 + \dots + a_k \cdot v_k)$$

$$= a_1 (c \cdot v_1) + a_2 (c \cdot v_2) + \dots + a_k (c \cdot v_k)$$

$$= a_1 \cdot 0 + a_2 \cdot 0 + \dots + a_k \cdot 0 = 0.$$

Since the choice of u is arbitrary,

$c \cdot u = 0$ for all $u \in \text{span}(S)$.

"$\Leftarrow$" This is immediate because $S \subseteq \text{span}(S)$.

6.

$Ac = b$ is equivalent to the equation

$$c_1 \begin{pmatrix} | \\ v_1 \\ | \end{pmatrix} + c_2 \begin{pmatrix} | \\ v_2 \\ | \end{pmatrix} + \cdots + c_n \begin{pmatrix} | \\ v_n \\ | \end{pmatrix} = \begin{pmatrix} | \\ b \\ | \end{pmatrix}$$

By definition $b \in \mathrm{span}(S)$ if and only if the solution for $c_1, \ldots, c_n$ exists.

7. $Ac = b$ is equivalent to

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 .$$

By definition, S is linearly independent

$\Leftrightarrow$ $c_1 = c_2 = \cdots = c_n = 0$ is the unique solution.