Sample Questions 8

1. Determine whether S is linearly independent or not.

   (a) $S = \left\{ \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 14 \end{bmatrix} \right\}$

   (b) $S = \left\{ \begin{bmatrix} 1 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix} \right\}$

   (c) $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}$

2. Consider the vector space of all functions on $\mathbb{R}$. Determine whether S is linearly independent or not.

   (a) $S = \{\cos(x), \sin(x)\}$

   (b) $S = \{1, \sin(x), \sin(2x)\}$

   (c) $S = \{1, \cos^2(x), \sin^2(x)\}$

   (d) $S = \{\cos(2x), \cos^2(x), \sin^2(x)\}$

3. Show that any $n+1$ vectors in $\mathbb{R}^n$ form a linearly dependent set.

4. Suppose S is a linearly independent set. Show that $S \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin \operatorname{span}(S)$.

5. Show that any superset of a linearly dependent set is linearly dependent. Also show that any subset of a linearly independent set is linearly independent.

6. Recall that two vectors $\mathbf{v}$ and $\mathbf{u}$ are orthogonal if $\mathbf{v} \cdot \mathbf{u} = 0$. Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of nonzero vectors such that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for any i and j. Show that S is linearly independent.

7. Let $\mathbf{A} = \left[a_{i,j}\right]$ be an $n \times n$ matrix such that $a_{i,i} \neq 0$ for all $i = 1, \ldots, n$ and $a_{i,j} = 0$ for all $i > j$. That is, A is an upper triangular matrix with all diagonal entries nonzero. Show that the columns of A form a linearly independent set. Similarly, show that the rows of A form a linearly independent set.