Sample Solutions for Sample Questions 9

1 (a). $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$

$E_1 \quad , \quad E_2 \quad , \quad E_3 \quad , \quad E_4.$

Note that

$$\text{all matrices} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}.$$

$$= \{ aE_1 + bE_2 + cE_3 + dE_4 \mid a, b, c, d \in \mathbb{R} \}.$$

$$= \text{span}(\{E_1, E_2, E_3, E_4\}).$$

And they are indep.

(b) $\langle 1, x, x^2, x^3 \rangle$

$$\text{all polynomials of deg} \leq 3 = \{ a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}^2 \}$$

$$= \text{span}(\{1, x, x^2, x^3\}).$$

And they are indep.

(c). Solve $c-2\underbrace{b}_{\text{free}}=0$ first. Note a is also a free variable.

Let $b=s$.
$a=t$
$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} t \\ s \\ 2s \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

So the set $= \left\{ t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + s\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$.

$= \text{span}\left(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\}\right)$.

They are indep.

So $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\rangle$ is a basis.

(d) $$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ so $\langle \vec{v}_1, \vec{v}_3 \rangle$ is a basis [make sure you understand why?]

$$\langle \vec{v}_1, \vec{v}_3 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

(e). $$\begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 1 & 1 & 8 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \langle \vec{v}_1, \vec{v}_2, \vec{v}_4 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \right\rangle$ is a basis.

(f)

$$\begin{pmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

free variables.

Solve $\beta_1, \beta_2, \beta_3$.

β_1: Let $x_2=1$, $x_3=0$, $x_4=0$ $\Rightarrow \begin{pmatrix} +4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

β_2: Let $x_2=0$, $x_3=1$, $x_4=0$ $\Rightarrow \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

β_3: Let $x_2=0$, $x_3=0$, $x_4=1$ $\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$\Rightarrow \left\langle \begin{pmatrix} +4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ is a basis for the solution set.

(Nov 11 更正)

注意！

(a),(b)　(c),(f)　(d),(e) 各為不同類型.

(a),(b)：找常見空間的基底。各個空間有個標準答案
e.g. $\mathbb{R}^n$ 有標準基底.

(c),(f)：找解空間的基底。用 $\beta_1, \beta_2, \ldots \beta_k$，其中 k 是自由變數的個數.

(d),(e)：找一個 span 的基底。用對應到領導變數的那些 column.

2.

① Claim: $\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle$ is a basis when $c_1, c_2, c_3 \neq 0$.

~~Since~~ $\cancel{c_1\vec{x}_1 \in \text{span}\langle x_1, x_2, x_3 \rangle}$

$\cancel{c_2\vec{x}_2}$

$\cancel{c_3\vec{x}_3}$

Note $\quad \vec{x}_1 = \frac{1}{c_1}\cdot(c_1\vec{x}_1) \in \text{span}\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle$.

Similarly $\vec{x}_2, \vec{x}_3 \in \text{span}\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle$.

So $\quad \text{span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle \subseteq \text{span}\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle$.

Also, $\cancel{c_1 x_1 \in x_2}$ $c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \in \text{span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$

so $\quad \text{span}\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle \subseteq \text{span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$.

In conclusion, $\text{span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle = \text{span}\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle$.

Now $\quad \text{span}\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle = \underbrace{\text{span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle}_{\text{3-dimensional space}}$

$\Rightarrow \underbrace{\langle c_1\vec{x}_1, c_2\vec{x}_2, c_3\vec{x}_3 \rangle}_{\text{also three vectors}}$ is a basis.

② Claim: $\langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$ is a basis.

Similarly,

$$\underbrace{\begin{aligned} \vec{y}_1 &= \vec{x}_1 + \vec{x}_1 \\ \vec{y}_2 &= \vec{x}_1 + \vec{x}_2 \\ \vec{y}_3 &= \vec{x}_1 + \vec{x}_3 \end{aligned}}_{\text{all in span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle} \qquad \underbrace{\begin{aligned} \vec{x}_1 &= \tfrac{1}{2}\vec{y}_1 \\ \vec{x}_2 &= \vec{y}_2 - \tfrac{1}{2}\vec{y}_1 \\ \vec{x}_3 &= \vec{y}_3 - \tfrac{1}{2}\vec{y}_1 \end{aligned}}_{\text{all in span}\langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle}$$

So $\quad \text{span}\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle = \text{span}\langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$.

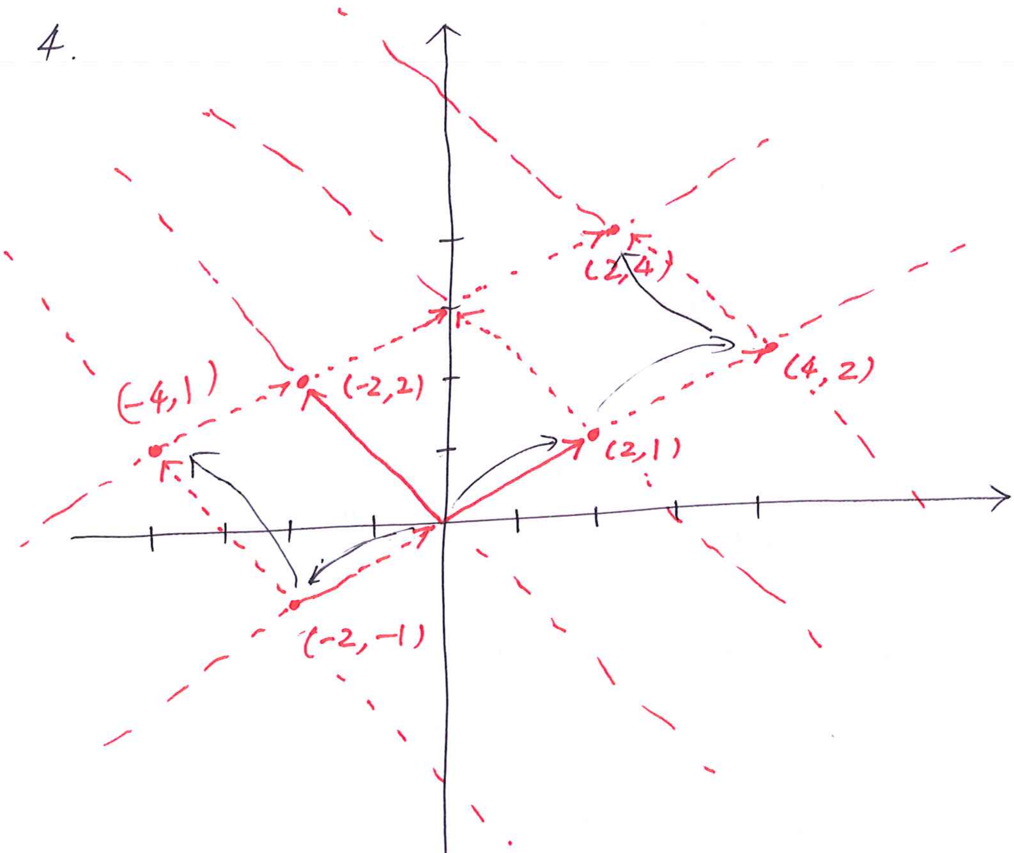
Since $\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$ is a basis and the spanning space has dimension 3, and $\langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$ has 3 vectors spanning the same space $\Rightarrow \langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$ is a basis.

3.

$$\text{all symmetric}^{3\times 3}\text{ matrices} = \left\{ \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \,\middle|\, a, b, c, d, e, f \in \mathbb{R} \right\}.$$

$$\Rightarrow \text{basis} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$$

4.



So $\mathrm{Rep}_B\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

meaning $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\mathrm{Rep}_B\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

meaning $\begin{pmatrix} -4 \\ 1 \end{pmatrix} = -1\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

5.

[Key ; find $\vec{v}$ not in the spanning set .]

(a). Find $\vec{v} \notin \operatorname{span}\left(\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}\right)$.

e.g. $\vec{v}$ can be $\begin{pmatrix}1\\0\end{pmatrix}$.

(b). Find $\vec{v} \notin \operatorname{span}\left(\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}\right)$.

e.g. $\vec{v}$ can be $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

(c). Find $\vec{v} \in \operatorname{span}\left(\left\{ x, 1+x^2 \right\}\right)$.

e.g. $\vec{v}$ can be x^2.

6. Suppose $x_{f_1}, x_{f_2}, \ldots, x_{f_k}$ be the k free variables.

So we set $x_{f_i} = 1$ and $x_{f_j} = 0$ for all $j \neq i$ to find $\vec{\beta_i}$.

This means the f_i-entry of $\vec{\beta_j}$ is 0 when $j \neq i$

and is 1 when $j = i$.

~~Let $(\beta_j)_i$ be the i-th entry of~~

Use the notation $(\vec{v})_i$ to denote the i-th entry of $\vec{v}$.

Suppose $c_1\vec{\beta_1} + \cdots + c_k\vec{\beta_k} = 0$.

$$\Rightarrow \underbrace{(c_1\vec{\beta_1})_{f_1}}_{=c_1} + \underbrace{\cdots}_{=0\ \ =0} + \underbrace{(c_k\vec{\beta_k})_{f_1}}_{=0} = 0 \Rightarrow c_1 = 0.$$

Similarly,
$$\underbrace{(c_1\vec{\beta_1})_{f_i}}_{=0} + \cdots + \underbrace{(c_i\vec{\beta_i})_{f_i}}_{=c_i} + \cdots + \underbrace{(c_k\vec{\beta_k})_{f_i}}_{=0} = 0 \Rightarrow c_i = 0.$$

$\Rightarrow c_1 = \cdots = c_k = 0 \Rightarrow S$ is linearly indep.