國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

期末考 June 17, 2019 **Final Examination**

姓名 Name : Solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	9 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Let $\mathcal{M}_{2\times 2}$ be the space of all 2×2 matrices. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times 2}$. Find a basis of the null space of f ~~and a basis of the range of f~~.

Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis of $\mathcal{M}_{2\times 2}$.

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 0 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix} \Rightarrow \overset{F}{\overset{\shortparallel}{Rep_{\mathcal{B},\mathcal{B}}(f)}} = \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \\ -3 & 0 & 9 & 0 \\ 0 & -3 & 0 & 9 \end{pmatrix}.$$

$$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 9 & 0 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} -3 \\ 0 \\ 9 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 0 & 9 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 0 \\ -3 \\ 0 \\ 9 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \cancel{\Rightarrow \text{Colspace}} \quad \Rightarrow \text{nullspace}(F) = \text{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

leading free

$$\Rightarrow \text{nullspace}(f) = \text{span}\underbrace{\left\{ \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \right\}}_{\text{basis}}.$$

2. Let $\mathbf{L}_n$ be the $n \times n$ matrix whose i, j-entry is -2 if $i = j$, 1 if $|i - j| = 1$, and 0 otherwise. For example,

$$\mathbf{L}_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \mathbf{L}_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \text{ and } \mathbf{L}_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(a) [1pt] Compute $\det(\mathbf{L}_n)$ for $n = 2, 3$.

$\det(L_2) = 4 - 1 = 3.$

$\det(L_3) = -8 + 2 + 2 = -4.$

(b) [2pt] Find a recurrence relation for $\det(\mathbf{L}_n)$. For example, find a and b such that

$$\det(\mathbf{L}_n) = a \det(\mathbf{L}_{n-1}) + b \det(\mathbf{L}_{n-2})_* .$$

~~where a and b may depend on n.~~

$\det(L_n) = \det \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$

$= -2 \det(L_{n-1}) - 1 \cdot \det \begin{pmatrix} 1 & 1 \; \cdots \\ \vdots & L_{n-2} \end{pmatrix}$

$= -2 \det(L_{n-1}) - \det(L_{n-2})$

$\det(L_n) = \det \begin{pmatrix} -1 & 1 & & & \\ 0 & -2 & \ddots & & \\ \vdots & 1 & \ddots & \ddots & 1 \\ -1 & & & 1 & -2 \end{pmatrix}$

[add every column to 1st col].

$= -1 \det(L_{n-1}) + (-1)^n$

(c) [2pt] ~~Find~~ Compute $\det(\mathbf{L}_n)$ ~~when~~ for $n = 5,$ 10.

$\det(L_4) = -2\det(L_3) - \det(L_2) = 5$

$\det(L_5) = -2\det(L_4) - \det(L_3) = \boxed{-6}$

$\vdots$

$\det(L_{10}) = -2\det(L_9) - \det(L_8) = \boxed{11}$

$x^2 \quad -x \quad +2$

$x^2 \quad -x \quad +2$

3. Let
$$m(x) = x^4 - 2x^3 + 5x^2 - 4x + 4.$$

(a) [1pt] Find the derivative $m'(x)$ of $m(x)$.

$m'(x) = 4x^3 - 6x^2 + 10x - 4.$

(b) [2pt] Find the Sylvester matrix $S_{m,m'}$ of $m(x)$ and $m'(x)$.

deg m + deg m' = 4+3 = 7

$$S_{m,m'} = \begin{pmatrix} 4 & 0 & 6 & -4 & 0 & 0 & 0 \\ -4 & 4 & \cancel{6}0 & 10 & -4 & 0 & 0 \\ 5 & -4 & 4 & -6 & 10 & -4 & 0 \\ -2 & 5 & -4 & 4 & -6 & 10 & -4 \\ 1 & -2 & 5 & 0 & 4 & -6 & 10 \\ 0 & 1 & -2 & 0 & 0 & 4 & -6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \end{pmatrix}$$

(c) [1pt] Recall that the resultant $\text{Res}(m, m') = \det(S_{m,m'})$ is the determinant of the Sylvester matrix. Describe how to tell if $m(x)$ and $m'(x)$ have a common root in $\mathbb{C}$ or not by the value of $\text{Res}(m, m')$.

$\text{Res}(m, m') = 0 \Leftrightarrow$ $m(x)$ and $m'(x)$ have a common root in $\mathbb{C}$.

(d) [1pt] Describe how to tell if $m(x)$ has a multiple root in $\mathbb{C}$ or not by the value of $\text{Res}(m, m')$.

$\text{Res}(m, m') = 0 \Leftrightarrow$ $m(x)$ has a multiple root in $\mathbb{C}$.

4. [5pt] Diagonalize

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

or show that it is not diagonalizable. (Note: The eigenvalues are integers.)

To diagonalize a matrix A, you have to find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$S_0 = 1$

$S_1 = 0+0+0+0 = 0$

$S_2 = 0-1-1-1-1+0 = -4$

$S_3 = 0+0+0+0 = 0$

$S_4 = 0$

$\Rightarrow$ char poly $p(x) = x^4 - 4x^2 = x^2(x+2)(x-2)$

$\Rightarrow$ eigenvalues : $0, 0, 2, -2$.

① $\lambda = 0 \quad A - \lambda I = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & & \\ & & 1 & 1 \\ & & & \\ & & & \end{pmatrix}$

$\Rightarrow E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$.

② $\lambda = 2, \quad A - 2I = \begin{pmatrix} -2 & & 1 & 1 \\ & -2 & 1 & 1 \\ 1 & 1 & -2 & \\ 1 & 1 & & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & & 1 & 1 \\ & -2 & 1 & 1 \\ & & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

③ $\lambda = -2, \quad A + 2I = \begin{pmatrix} 2 & & 1 & 1 \\ & 2 & 1 & 1 \\ 1 & 1 & 2 & \\ 1 & 1 & & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & & 1 & 1 \\ & 2 & 1 & 1 \\ & & 1 & -1 \\ & & & \end{pmatrix}$

$\Rightarrow E_{-2} = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$Q = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & -2 \end{pmatrix} \Rightarrow AQ = QD \Leftrightarrow Q^{-1}AQ = D.$

5. [2pt] Is the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonalizable or not diagonalizable? Justify your answer.

A is diagonalizable

since all eigenvalues $4, 3, 2, 1$ are distinct.

6. [3pt] Is the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

diagonalizable or not diagonalizable? Justify your answer.

A is not diagonalizable

since alg mult $= 4$ for the eigenvalue 4.
geo mult $= 1$

char poly of A is $p(x) = (x-4)^4$.

$\Rightarrow$ alg mult $= 4$.

$$A - 4I = \begin{pmatrix} 0 & 3 & 2 & 1 \\ & 0 & 3 & 2 \\ & & 0 & 3 \\ & & & 0 \end{pmatrix}$$

$\Rightarrow$ geo mult $= \dim E_4 = 1$.

7. [1pt] What is the definition of "matrix **A** is similar to matrix **B**"?

A is similar to B if

there is an invertible matrix Q such that $Q^{-1}AQ = B$.

8. [1pt] Suppose A is invertible and $\det(\mathbf{A}) \neq 0$ is known. How to obtain $\det(\mathbf{A}^{-1})$ from $\det(\mathbf{A})$?

$\det(A^{-1}) = \det(A)^{-1}$.

9. [~~4~~ 3pt] Show that similar matrices have the same characteristic polynomial.

Suppose A and B are similar.

$\Rightarrow$ exist an invertible matrix Q

such that $Q^{-1}AQ = B$.

char poly of A $= \det(A - xI)$

$= \det(Q^{-1}) \cdot \det(A - xI) \cdot \det(Q)$

$= \det(Q^{-1}(A - xI)Q)$ ~~det~~

$= \det(Q^{-1}AQ - xQ^{-1}IQ)$

$= \det(B - xI)$

$=$ char poly of B.

10. [1pt] Describe the Cayley–Hamilton theorem.

If A is a square matrix and $p(x)$ is its char poly, then $p(A) = 0$, the zero matrix.

11. [2pt] Show that the Cayley–Hamilton theorem is true for

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

char poly of $A = (x-4)(x-3)(x-2)(x-1) = p(x)$

$\Rightarrow p(A) = (A-4I)(A-3I)(A-2I)(A-I)$.

$$= \begin{pmatrix} 0 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 4 & & & \\ & 0 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \cancel{0}\,4 & & & \\ & 3 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 0 \end{pmatrix}$$

$= 0$.

12. [2pt] Suppose the Cayley–Hamilton theorem is true for diagonal matrices. Show that the Cayley–Hamilton theorem is true for any diagonalizable matrices.

Suppose A is diagonalizable.

$\Rightarrow$ exist invertible matrix Q such that $Q^{-1}AQ = D$.
diagonal matrix D

Let $p(x)$ be the char poly of A.

$\Rightarrow p(x)$ is also char poly of D.

By assumption, $p(D) = 0$.

$\Rightarrow p(Q^{-1}AQ) = 0$

$\Rightarrow Q^{-1}p(A)Q = 0$

$\Rightarrow P(A) = Q\,0\,Q^{-1} = 0$.

13. [extra 5pt] Find the characteristic polynomial $p(x)$ for the matrix $\mathbf{J}_n - \mathbf{I}_n$. Here $\mathbf{J}_n$ is the $n \times n$ all-ones matrix and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

For J_n,

$$S_0 = 1$$

$$S_1 = n \quad \cancel{\Rightarrow \text{char poly of } J_n}$$

$$S_2 = S_3 = \dots = S_n = 0$$

$$\Rightarrow \text{char poly of } J_n = \det(J_n - xI)$$

$$= \underbrace{(-1)^n\left[x^n - n x^{n-1}\right]}_{q(x)} = (-1)^n x^{n-1}(x-n)$$

That, char poly of $J_n - I_n = \det(J_n - I_n - xI_n)$

$$= \det(J_n - (x+1)I_n)$$

$$= q(x+1) = (-1)^n (x+1)^{n-1}(x+1-n)$$

$$\cancel{= (-1)^n\left[(x+1)^n\right.}$$

14. Find the minimal polynomial of the matrix

size 2

$$\mathbf{A} = \begin{bmatrix} \boxed{\begin{matrix}1 & 1\\ 0 & 1\end{matrix}} & 0&0&0&0&0&0&0&0 \\ 0&0&\boxed{\begin{matrix}1 & 1\\ 0 & 1\end{matrix}}&0&0&0&0&0&0 \\ 0&0&0&0&\boxed{\begin{matrix}2&1&0\\0&2&1\\0&0&2\end{matrix}}&0&0&0 \\ 0&0&0&0&0&0&0&\boxed{\begin{matrix}2&1\\0&2\end{matrix}}&0 \\ 0&0&0&0&0&0&0&0&0&\boxed{3} \end{bmatrix}$$

max size 3.

size 1.

(You do not have to justify your answer.)

$m(x) = (x-1)^2 (x-2)^3 (x-3)$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	2	
Total	35 (+7)	