國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

期末考 June 17, 2019 **Final Examination**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	9 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

– Enter your **Name** and **Student ID #** before you start.

– Using the calculator is not allowed (and not necessary) for this exam.

– Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.

– Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

– 可用中文或英文作答

1. [5pt] Let $\mathcal{M}_{2\times 2}$ be the space of all 2×2 matrices. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times 2}$. Find a basis of the null space of f.

2. Let $\mathbf{L}_n$ be the $n \times n$ matrix whose i, j-entry is -2 if $i = j$, 1 if $|i - j| = 1$, and 0 otherwise. For example,

$$\mathbf{L}_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \mathbf{L}_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \text{ and } \mathbf{L}_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(a) [1pt] Compute $\det(\mathbf{L}_n)$ for $n = 2, 3$.

(b) [2pt] Find a recurrence relation for $\det(\mathbf{L}_n)$. For example, find a and b such that

$$\det(\mathbf{L}_n) = a \det(\mathbf{L}_{n-1}) + b \det(\mathbf{L}_{n-2}).$$

(c) [2pt] Compute $\det(\mathbf{L}_n)$ for $n = 5, 10$.

3. Let
$$m(x) = x^4 + 2x^3 + 5x^2 + 4x + 4.$$

(a) [1pt] Find the derivative $m'(x)$ of $m(x)$.

(b) [2pt] Find the Sylvester matrix $S_{m,m'}$ of $m(x)$ and $m'(x)$.

(c) [1pt] Recall that the resultant $\mathrm{Res}(m, m') = \det(S_{m,m'})$ is the determinant of the Sylvester matrix. Describe how to tell if $m(x)$ and $m'(x)$ have a common root in $\mathbb{C}$ or not by the value of $\mathrm{Res}(m, m')$.

(d) [1pt] Describe how to tell if $m(x)$ has a multiple root in $\mathbb{C}$ or not by the value of $\mathrm{Res}(m, m')$.

4. [5pt] Diagonalize

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

or show that it is not diagonalizable. To diagonalized a matrix $\mathbf{A}$, you have to find an invertible matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$. (Note: For this problem, the eigenvalues are integers.)

5. [2pt] Is the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

diagonalizable or not diagonalizable? Justify your answer.

6. [3pt] Is the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonalizable or not diagonalizable? Justify your answer.

7. [1pt] What is the definition of "matrix $\mathbf{A}$ is similar to matrix $\mathbf{B}$"?

8. [1pt] Suppose A is invertible and $\det(\mathbf{A}) \neq 0$ is known. How to obtain $\det(\mathbf{A}^{-1})$ from $\det(\mathbf{A})$?

9. [3pt] Show that similar matrices have the same characteristic polynomial.

10. [1pt] Describe the Cayley–Hamilton theorem.

11. [2pt] Show that the Cayley–Hamilton theorem is true for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

12. [2pt] Suppose the Cayley–Hamilton theorem is true for diagonal matrices. Show that the Cayley–Hamilton theorem is true for any diagonalizable matrices.

13. [extra 5pt] Find the characteristic polynomial $p(x)$ for the matrix $\mathbf{J}_n - \mathbf{I}_n$. Here $\mathbf{J}_n$ is the $n \times n$ all-ones matrix and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

14. [extra 2pt] Find the minimal polynomial of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

(You do not have to justify your answer.)

[END]

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4	5	
5	5	
6	5	
7	5	
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