國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

期末考 June 17, 2019 **Final Examination**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **9 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Let $\mathcal{M}_{2\times 2}$ be the space of all 2×2 matrices. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times 2}$. Find a basis of the null space of f ~~and a basis of the range of f~~.

Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis of $\mathcal{M}_{2\times 2}$.

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} \xrightarrow{\operatorname{Rep}_{\mathcal{B}}} \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \xrightarrow{\operatorname{Rep}_{\mathcal{B}}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} \Rightarrow \operatorname{Rep}_{\mathcal{B},\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -2 & 0 & 4 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix} = F.$$

$$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 4 & 0 \end{pmatrix} \xrightarrow{\operatorname{Rep}_{\mathcal{B}}} \begin{pmatrix} -2 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 4 \end{pmatrix} \xrightarrow{\operatorname{Rep}_{\mathcal{B}}} \begin{pmatrix} 0 \\ -2 \\ 0 \\ 4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{nullspace}(F) = \text{span}\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\Rightarrow \text{nullspace}(f) = \text{span} \underbrace{\left\{ \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \right\}}_{\text{basis}}.$$

2. Let $\mathbf{L}_n$ be the $n \times n$ matrix whose i, j-entry is -2 if $i = j$, 1 if $|i - j| = 1$, and 0 otherwise. For example,

$$\mathbf{L}_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \mathbf{L}_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \text{ and } \mathbf{L}_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(a) [1pt] Compute $\det(\mathbf{L}_n)$ for $n = 2, 3$.

See ver. A.

(b) [2pt] Find a recurrence relation for $\det(\mathbf{L}_n)$. For example, find a and b such that

$$\det(\mathbf{L}_n) = a \det(\mathbf{L}_{n-1}) + b \det(\mathbf{L}_{n-2}),$$

where a and b may depend on n.

See ver. A.

(c) [2pt] Find $\det(\mathbf{L}_n)$ when $n = 10$.

See ver. A.

3. Let

$$m(x) = x^4 + 2x^3 + 5x^2 + 4x + 4.$$

$x^2 + x + 2$
$x^2 + x + 2$

(a) [1pt] Find the derivative $m'(x)$ of $m(x)$.

$m'(x) = 4x^3 + 6x^2 + 10x + 4$

(b) [2pt] Find the Sylvester matrix $S_{m,m'}$ of $m(x)$ and $m'(x)$.

$\deg m + \deg m' = 4 + 3 = 7.$

$$S_{m,m'} = \begin{pmatrix} 4 & 0 & 0 & 4 & 0 & 0 & 0 \\ 4 & 4 & 0 & 10 & 4 & 0 & 0 \\ 5 & 4 & 4 & 6 & 10 & 4 & 0 \\ 2 & 5 & 4 & 4 & 6 & 10 & 4 \\ 1 & 2 & 5 & 0 & 4 & 6 & 10 \\ 0 & 1 & 2 & 0 & 0 & 4 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

(c) [1pt] Recall that the resultant $\text{Res}(m, m') = \det(S_{m,m'})$ is the determinant of the Sylvester matrix. Describe how to tell if $m(x)$ and $m'(x)$ have a common root in $\mathbb{C}$ or not by the value of $\text{Res}(m, m')$.

See ver. A

(d) [1pt] Describe how to tell if $m(x)$ has a multiple root in $\mathbb{C}$ or not by the value of $\text{Res}(m, m')$.

See ver. A .

4. [5pt] Diagonalize

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

or show that it is not diagonalizable. (Note: The eigenvalues are integers.)

~~By direct computa~~

$A\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 2 \Rightarrow \lambda=2$ is an eigval.

nullity $A = 2 \Rightarrow \lambda=0$ is an eigval with geo mult ≥ 2.

~~Since~~ Suppose char poly of A

is $p(x) = (x-2)(x)(x)(x-\lambda_4)$.

$\Rightarrow 2+0+0+\lambda_4 = \operatorname{tr}(A) = 0 \Rightarrow \lambda_4 = -2$.

Thus, $p(x) = x^2(x+2)(x-2) \Rightarrow \lambda = 0, 0, 2, -2$.

① $\lambda = 0$, $A - 0I = \begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$

$\Rightarrow E_0 = \operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}\right\}$.

② $\lambda = 2$, ~~$A-2I=$~~ we already knew $E_2 = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$.

③ $\lambda = -2$, solve $E_{-2} = \operatorname{Span}\left\{\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\right\}$.

$\Rightarrow Q = \begin{pmatrix} -1&0&1&-1 \\ 0&-1&1&1 \\ 1&0&1&-1 \\ 0&1&1&1 \end{pmatrix}$, $D = \begin{pmatrix} 0&&& \\ &0&& \\ &&2& \\ &&&-2 \end{pmatrix}$ and $AQ = QD$

$\Updownarrow$

$Q^{-1}AQ = D$.

5. [2pt] Is the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

diagonalizable or not diagonalizable? Justify your answer.

A is diagonalizable

since all eigenvalues are distinct.
1, 2, 3, 4

6. [3pt] Is the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonalizable or not diagonalizable? Justify your answer.

A is not diagonalizable

since for $\lambda = 1$, alg mult $= 4$
geo mult $= 1$.

char poly of A is $p(x) = (x-1)^4$

$\Rightarrow$ alg mult $= 4$.

$$A - I = \begin{pmatrix} 0 & 2 & 3 & 4 \\ & 0 & 2 & 3 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix} \Rightarrow \text{geo mult} = \dim E_1 = 1.$$

7. [1pt] What is the definition of "matrix $\mathbf{A}$ is similar to matrix $\mathbf{B}$"?

see ver. A

8. [1pt] Suppose A is invertible and $\det(\mathbf{A}) \neq 0$ is known. How to obtain $\det(\mathbf{A}^{-1})$ from $\det(\mathbf{A})$?

see ver. A

9. [4pt] Show that similar matrices have the same characteristic polynomial.

see ver. A.

10. [1pt] Describe the Cayley–Hamilton theorem.

see ver. A .

11. [2pt] Show that the Cayley–Hamilton theorem is true for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

See ver. A .

12. [2pt] Suppose the Cayley–Hamilton theorem is true for diagonal matrices. Show that the Cayley–Hamilton theorem is true for any diagonalizable matrices.

see ver. A .

13. [extra 5pt] Find the characteristic polynomial $p(x)$ for the matrix $\mathbf{J}_n - \mathbf{I}_n$. Here $\mathbf{J}_n$ is the $n \times n$ all-ones matrix and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

See ver. A.

14. Find the minimal polynomial of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

(You do not have to justify your answer.)

see ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	2	
Total	35 (+7)	