國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） MATH 104 / GEAI 1209: Linear Algebra II

第一次期中考 March 25, 2019 **Midterm 1**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	7 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [2pt] Let V and W be two vector spaces. Let $f : V \to W$ be a function. Write down the definition of f being a homomorphism.

2. Let $\mathcal{P}_2$ be the space of all polynomials with degree at most 2. Let

$$\mathcal{B} = \{1, x+2, (x+2)^2\}$$

be a basis of $\mathcal{P}_2$.

(a) [2pt] Find the representation $\mathrm{Rep}_{\mathcal{B}}(x^2)$.

(b) [1pt] Suppose $\mathbf{v}$ is a vector in $\mathcal{P}_2$ with $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v}) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. Find $\mathbf{v}$. (You do not have to expand the your answer.)

3. [5pt] Let $\mathcal{M}_{2\times 2}$ be the space of all 2×2 matrices. Let $\mathbf{E}_{i,j}$ be the 2×2 matrix whose i, j-entry is 1 and other entries are zeros. Then

$$\mathcal{B} = \{\mathbf{E}_{1,1}, \mathbf{E}_{1,2}, \mathbf{E}_{2,1}, \mathbf{E}_{2,2}\}$$

is a basis of $\mathcal{M}_{2\times 2}$. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times 2}$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

4. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

(a) [2pt] Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be homomorphism such that

$$f(\mathbf{v}_1) = f(\mathbf{v}_2) = f(\mathbf{v}_3) = \mathbf{u}_1.$$

Find $f(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$.

(b) [3pt] Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$ and let $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$. Suppose $g : \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism with $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(g) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \end{bmatrix}$. Find $g(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$.

5. [5pt] Let $\mathcal{M}_{2\times 2}$ be the space of all 2×2 matrices. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times 2}$. Find a basis of the null space of f and a basis of the range of f. [If you use your answer from Problem 3, double check your answer to make sure it is correct.]

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of $\mathbb{R}^2$ and let $\mathcal{S}_2$ be the standard basis of $\mathbb{R}^2$.

   (a) [2pt] Find $\mathrm{Rep}_{\mathcal{S}_2,\mathcal{B}}(\mathrm{id})$, the change of basis matrix from $\mathcal{S}_2$ to $\mathcal{B}$.

   (b) [3pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homomorphism with $\mathrm{Rep}_{\mathcal{S}_2,\mathcal{S}_2}(f) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

7. [5pt] Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \end{bmatrix}$. Find matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{PAQ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

8. [extra 2pt] Define a sequence by $a_0 = 6$, $a_1 = 13$, and

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

for all $n \geq 2$. Find a formula for a_n.
[Hints: First make an observation that

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

Then you may use the fact

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1}$$

to find the answer.]

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	2	
Total	30 (+2)	