國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第一次期中考 March 25, 2019 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **7 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [2pt] Let V and W be two vector spaces. Let $f : V \to W$ be a function. Write down the definition of f being a homomorphism.

$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ for all $\vec{v}_1, \vec{v}_2 \in V$

$f(r\vec{v}) = r \cdot f(\vec{v})$ for all $r \in \mathbb{R}$, $\vec{v} \in V$.

2. Let $\mathcal{P}_2$ be the space of all polynomials with degree at most 2. Let

$$\mathcal{B} = \{1, x+2, (x+2)^2\}$$

be a basis of $\mathcal{P}_2$.

(a) [2pt] Find the representation $\mathrm{Rep}_{\mathcal{B}}(x^2)$.

$1 = 1$

$x+2 = 2 + x$

$(x+2)^2 = 4 + 4x + x^2$

$x^2 = 4 \cdot 1 - 4(x+2) + 1 \cdot (x+2)^2$

$\Rightarrow \mathrm{Rep}_{\mathcal{B}}(x^2) = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$

(b) [1pt] Suppose $\mathbf{v}$ is a vector in $\mathcal{P}_2$ with $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v}) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. Find $\mathbf{v}$. (You do not have to expand the your answer.)

$\vec{v} = 2 \cdot 1 + 4 \cdot (x+2) + 6 \cdot (x+2)^2$.

3. [5pt] Let $\mathcal{M}_{2\times2}$ be the space of all 2×2 matrices. Let $\mathbf{E}_{i,j}$ be the 2×2 matrix whose i, j-entry is 1 and other entries are zeros. Then

$$\mathcal{B} = \{\mathbf{E}_{1,1}, \mathbf{E}_{1,2}, \mathbf{E}_{2,1}, \mathbf{E}_{2,2}\}$$

is a basis of $\mathcal{M}_{2\times2}$. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times2} \to \mathcal{M}_{2\times2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times2}$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

$$f(E_{1,1}) = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

$$f(E_{1,2}) = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1\cancel{3} \\ 0 & 3\cancel{9} \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 0 \\ 1\cancel{3} \\ 0 \\ 3\cancel{9} \end{pmatrix}_{\mathcal{B}}.$$

$$f(E_{2,1}) = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 9 & 0 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 3 \\ 0 \\ 9 \\ 0 \end{pmatrix}_{\mathcal{B}}.$$

$$f(E_{2,2}) = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 9 \end{pmatrix} \xrightarrow{Rep_{\mathcal{B}}} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 9 \end{pmatrix}_{\mathcal{B}}.$$

$$\Rightarrow Rep_{\mathcal{B},\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 3 & 0 & 9 & 0 \\ 0 & 3 & 0 & 9 \end{pmatrix}$$

4. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

(a) [2pt] Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be homomorphism such that

$$f(\mathbf{v}_1) = f(\mathbf{v}_2) = f(\mathbf{v}_3) = \mathbf{u}_1.$$

Find $f(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f(1 \cdot \vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3) = 1 \cdot \vec{u}_1 - 3 \cdot \vec{u}_1 + 2\vec{u}_1 = 0 \cdot \vec{u}_1$$

$$= \begin{pmatrix} 0 \\ 0 \\ \cancel{0} \end{pmatrix}$$

(b) [3pt] Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$ and let $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis of $\mathbb{R}^2$. Suppose $g : \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism with $\text{Rep}_{\mathcal{B},\mathcal{D}}(g) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \end{bmatrix}$. Find $g(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$.

$$\text{Rep}_{\mathcal{B}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}_{\mathcal{B}} \text{ by (a)}.$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 9 & 0 \end{pmatrix}_{\mathcal{B},\mathcal{D}} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 2 \\ -26 \end{pmatrix}_{\mathcal{D}}.$$

$$\Rightarrow g\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 \cdot \vec{u}_1 - 26 \cdot \vec{u}_2 = \begin{pmatrix} 4 \\ 10 \end{pmatrix} - \begin{pmatrix} 130 \\ 52 \end{pmatrix}$$

$$= \begin{pmatrix} -126 \\ \cancel{10} \\ -42 \end{pmatrix}$$

5. [5pt] Let $\mathcal{M}_{2\times 2}$ be the space of all 2×2 matrices. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ and define the homomorphism $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by $f(\mathbf{M}) = \mathbf{AM}$ for all $\mathbf{M} \in \mathcal{M}_{2\times 2}$. Find ~~the~~ a basis of the null space and ~~the~~ a basis of the range of f. [If you use your answer from Problem 3, double check your answer to make sure it is correct.]

Let $\mathcal{B} = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$ be a basis of $\mathcal{M}_{2\times 2}$.

Then $\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 3 & 0 & 9 & 0 \\ 0 & 3 & 0 & 9 \end{pmatrix}$ by Problem 3.

Do row operations

$$\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(f) \rightsquigarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \uparrow$ free.

① $\Rightarrow \operatorname{Colspace}(\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(f)) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$

$\Rightarrow \operatorname{range}(f) = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}}_{} \right\}$

basis.

② $\operatorname{nullspace}(\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(f)) = \operatorname{span}\left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\Rightarrow \operatorname{nullspace}(f) = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -3 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ 0 & 1 \end{pmatrix}}_{\text{basis}} \right\}.$

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of $\mathbb{R}^2$ and let $\mathcal{S}_2$ be the standard basis of $\mathbb{R}^2$.

(a) [2pt] Find $\text{Rep}_{\mathcal{S}_2,\mathcal{B}}(\text{id})$, the change of basis matrix from $\mathcal{S}_2$ to $\mathcal{B}$.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \xrightarrow{\text{Rep}_{\mathcal{B}}} \begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix}_{\mathcal{B}}.$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{2}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \xrightarrow{\text{Rep}_{\mathcal{B}}} \begin{pmatrix} 2/5 \\ -1/5 \end{pmatrix}_{\mathcal{B}}.$$

$$\Rightarrow \text{Rep}_{S_2,\mathcal{B}}(\text{id}) = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}$$

(b) [3pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homomorphism with $\text{Rep}_{\mathcal{S}_2,\mathcal{S}_2} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find $\text{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

Easy to compute $\text{Rep}_{\mathcal{B}-S_2}(\text{id}) = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Then $\text{Rep}_{\mathcal{B}-\mathcal{B}}(f) = \text{Rep}_{S_2,\mathcal{B}}(\text{id}) \cdot \text{Rep}_{S_2,S_2}(f) \cdot \text{Rep}_{\mathcal{B},S_2}(\text{id})$

$$= \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$$

7. [5pt] Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \end{bmatrix}$. Find matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{PAQ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$A \xrightarrow{-2r_1+r_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[-3C_1+C_3]{2C_1+C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{C_2+C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$P = \overset{-2r_1+r_2}{\begin{pmatrix} 1 & 0 & \cancel{0} \\ -2 & 1 & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} \end{pmatrix}}$$

$$\cancel{Q = \underset{C_2+C_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \underset{\substack{2C_1+C_2 \\ -3C_1+C_3}}{\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

$$Q = \underset{\substack{2C_1+C_2 \\ -3C_1+C_3}}{\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{C_2+C_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

8. [extra 2pt] Define a sequence by $a_0 = 6$, $a_1 = 13$, and

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

for all $n \geq 2$. Find a formula for a_n.
[Hints: First make an observation that

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

Then you may use the fact

$$\begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1}$$

to find the answer.]

Try a few more :

$$a_2 = 5a_1 - 6a_0 = 5 \cdot 13 - 6 \cdot 6 = 29.$$

So $$\begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}.$$

$$\begin{pmatrix} 29 \\ 13 \end{pmatrix} \qquad \begin{pmatrix} 13 \\ 6 \end{pmatrix}$$

So $$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^n \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 2^n & 3 \cdot 3^n \\ 2^n & 3^n \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 2^n & 3 \cdot 3^n \\ 2^n & 3^n \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \cdot 2^n + 3 \cdot 3^n \\ 5 \cdot 2^n + 1 \cdot 3^n \end{pmatrix}$$

[END]

$$\Rightarrow a_n = 5 \cdot 2^n + 1 \cdot 3^n.$$

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	2	
Total	30 (+2)	