國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第二次期中考 May 6, 2019 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	8 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Let $V = \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a vector space. Use the Gram–Schmidt algorithm to find an **orthonormal** basis of V.

$$\boxed{\text{proj}_{[\vec{u}]}(\vec{v}) = \frac{\langle \vec{v}, \vec{u} \rangle}{|u|^2} \vec{u}}$$

Let $\vec{K}_1 = \vec{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{K}_2 = \vec{b}_2 - \text{proj}_{[K_1]}(\vec{b}_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{K}_3 = \vec{b}_3 - \text{proj}_{[K_1]}(\vec{b}_3) - \text{proj}_{[K_2]}(\vec{b}_3)$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \frac{1/2}{3/2}\begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{pmatrix}.$$

$\Rightarrow \{\vec{K}_1, \vec{K}_2, \vec{K}_3\}$ is orthogonal

$\Rightarrow \left\{\frac{1}{\sqrt{2}} \cdot \vec{K}_1, \frac{1}{\sqrt{3/2}} \vec{K}_2, \frac{1}{\sqrt{4/3}} \vec{K}_3\right\}$ is orthonormal.

2. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 10 & 10 & 10 \\ 1 & 3 & 10 & 11 & 11 \\ 1 & 3 & 10 & 11 & 12 \end{bmatrix}.$$

Find $\det(\mathbf{A})$.

$$\det(A) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 & 2 \\ & 2 & 9 & 9 & 9 \\ & 2 & 9 & 10 & 10 \\ & 2 & 9 & 10 & 11 \end{pmatrix} = \det\begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 9 & 9 & 9 \\ 2 & 9 & 10 & 10 \\ 2 & 9 & 10 & 11 \end{pmatrix}$$

$$= \det\begin{pmatrix} 2 & 2 & 2 & 2 \\ & 7 & 7 & 7 \\ & 7 & 8 & 8 \\ & 7 & 8 & 9 \end{pmatrix} = \det\begin{pmatrix} 2 & 2 & 2 & 2 \\ & 7 & 7 & 7 \\ & & 1 & 1 \\ & & 1 & 2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 2 & 2 & 2 & 2 \\ & 7 & 7 & 7 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} = \underline{\underline{14}}.$$

3. [2pt] Let

$$\mathbf{A} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}.$$

By the permutation expansion, det(**A**) is the sum of 24 terms. Find the signs of the following terms. [Write + or − in the blank.]

(a) (+)$dgjm$ $\longrightarrow$ $\phi = 4\ 3\ 2\ 1$ 2 switches

(b) (—)$bglm$ $\longrightarrow$ $\phi = 2\ 3\ 4\ 1$ 3 switches

(c) (+)$chin$ $\longrightarrow$ $\phi = 3\ 4\ 1\ 2$ 2 switches

(d) (—)$dejo$ $\longrightarrow$ $\phi = 4\ 1\ 2\ 3$ 3 switches

4. [1pt] Let **A** be a 5×5 matrix. According to the permutation expansion, how many terms are there in det(**A**)? Also, how many terms in det(**A**) ~~has~~ have negative signs?

- det(A) has $5! = \underline{120}$ terms.
- $\underline{60}$ of them have negative signs.

5. [2pt] Let

$$\mathbf{A} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & e & f & 0 \\ 0 & g & h & 0 \\ c & 0 & 0 & d \end{bmatrix}.$$

Find the formula of det(**A**).

$$\det(A) = a \cdot \det\begin{pmatrix} e & f & \\ g & h & \\ & & d \end{pmatrix} - b \cdot \det\begin{pmatrix} & e & f \\ & g & h \\ c & & \end{pmatrix}$$

$$= a \cdot (eh - gf) \cdot d - b \cdot (c) \cdot (eh - gf)$$

$$= (ad - bc) \cdot (eh - gf)$$

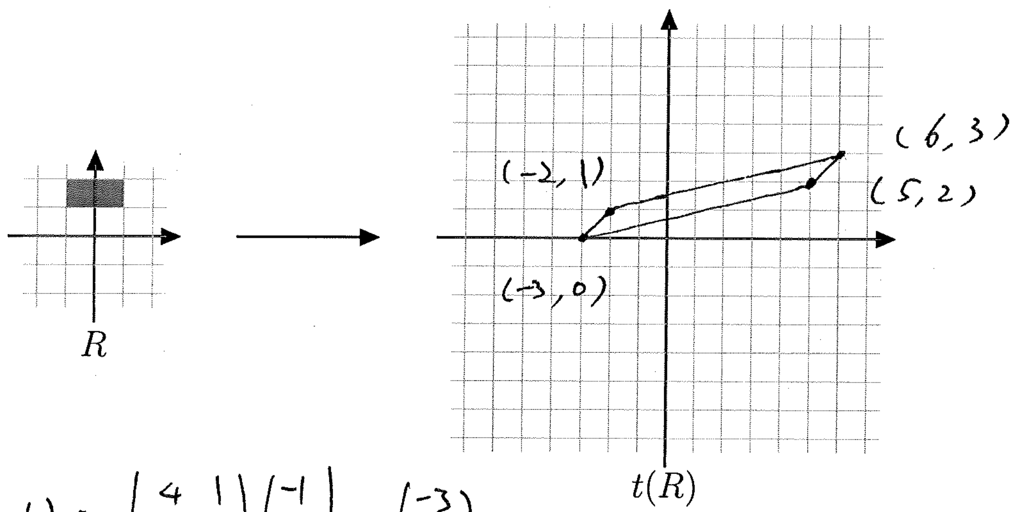
or

$$= \underline{adeh - adgf - bceh + bcgf.}$$

6. [5pt] Let R be the rectangle enclosed by the four vertices

$$(-1,1),(1,1),(1,2),(-1,2).$$

Let $t:\mathbb{R}^2\to\mathbb{R}^2$ be a homomorphism defined by $t(\mathbf{x})=\mathbf{T}\mathbf{x}$ with $\mathbf{T}=\begin{bmatrix}4&1\\1&1\end{bmatrix}$. Draw the region $t(R)$ and compute its area.



$$t(-1,1)=\begin{pmatrix}4&1\\1&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}$$

$$t(1,1)=\begin{pmatrix}5\\2\end{pmatrix}$$

$$t(1,2)=\begin{pmatrix}6\\3\end{pmatrix}$$

$$t(-1,2)=\begin{pmatrix}-2\\1\end{pmatrix}$$

Area $=\left|\det(T)\right|\cdot$ area of R

$=\ 3\cdot 2\ =\ \underline{\underline{6}}.$

7. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

(a) [1pt] Find $\det(\mathbf{A})$.

$$\det(A) = 5! = \underline{\underline{120}}.$$

(b) [2pt] Let $\mathbf{A}^{-1}$ be the inverse of $\mathbf{A}$. Find the 3, 2-entry of $\mathbf{A}^{-1}$.

$$3,2\text{-entry of } A^{-1} = [3,2\text{-entry of } \mathrm{adj}(A)] / \det(A)$$

$$= [2,3\text{-cofactor of } A] / 120.$$

$$= (-1)^5 \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix} / 120$$

$$= \underline{\underline{0}}.$$

(c) [2pt] Find the 2, 3-entry of $\mathbf{A}^{-1}$.

$$2,3\text{-entry of } A^{-1} = [3,2\text{-cofactor of } A] / \det(A)$$

$$= (-1)^5 \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 4 & 1 \\ & & & 5 \end{pmatrix} / \det(A)$$

$$= -20/120 = \underline{\underline{-\frac{1}{6}}}.$$

8. [5pt] Let $\mathbf{A}$ be an $n \times n$ matrix whose columns are $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent, then $\det(\mathbf{A}) = 0$.

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent,

there is a vector $\vec{v}_k$ that can be written as

$$\vec{v}_k = \sum_{i \neq k} c_i \vec{v}_i .$$

Apply row operations $\xrightarrow{-c_i r_i - r_k}$ for each $i \neq k$.

This does not change the determinant,

but the resulting matrix has a zero row.

$\Rightarrow \det(A) = 0$.

9. [5pt] Let $\mathbf{L}_n$ be the $n \times n$ matrix whose i, j-entry is -2 if $i = j$, 1 if $|i - j| = 1$, and 0 otherwise. For example,

$$\mathbf{L}_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \mathbf{L}_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \text{ and } \mathbf{L}_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Find the formula of $\det(\mathbf{L}_n)$ in terms of n. [You have to justify your answer.]

Ans: $\det(L_n) = \cancel{n+1}\ (-1)^n \cdot (n+1)$

Proof: By Induction.

Base step:

$\det(L_1) = -2$

$\det(L_2) = 4 - 1 = 3$

Hypothesis: $\det(L_n) = \cancel{n+1}\ (-1)^n \cdot (n+1)$ for small n.

Induction step:

Note that

$$\det(L_n) = \det\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} = -2 \cdot \det(L_{n-1}) - 1 \cdot \det\begin{pmatrix} 1 & 1 & & \\ & -2 & \ddots & \\ & 1 & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$$

$$= -2 \det(L_{n-1}) - \det(L_{n-2})$$

$$= -2(-1)^{n-1}(n) - (-1)^{n-2}(n-1) \quad \text{(by hypothesis)}$$

$$= (-1)^n(2n - n + 1) = (-1)^n(n+1).$$

10. [extra 2pt] Let $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$. Follow the instructions below to find an invertible matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$.

1. Compute the polynomial $p(t) = \det(\mathbf{A} - t\mathbf{I})$, where $\mathbf{I} = \mathbf{I}_2$ is the identity matrix.
2. Solve $p(t) = 0$ and get the two roots $\Lambda = \{\lambda_1, \lambda_2\}$.
3. For each $\lambda \in \Lambda$, compute a basis of the nullspace of $\mathbf{A} - \lambda\mathbf{I}$. (In this special case, say nullspace$(A - \lambda_1\mathbf{I}) = \text{span}\{\mathbf{v}_1\}$ and nullspace$(\mathbf{A} - \lambda_2\mathbf{I}) = \text{span}\{\mathbf{v}_2\}$.)
4. Let $\mathbf{Q}$ be the matrix whose columns are $\{\mathbf{v}_1, \mathbf{v}_2\}$. Then compute $\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$. (If everything works out, your $\mathbf{D}$ is a diagonal matrix.)

1. $p(t) = \det(A - tI) = \det\begin{pmatrix} 5-t & -6 \\ 1 & -t \end{pmatrix} = -t(5-t)+6$

2. If $p(t) = 0 \Rightarrow \cancel{t = 0, 5.}\ t^2 - 5t + 6 = 0 \Rightarrow t = 2, 3$

3. ① $\lambda = 2$. Solve $(A-2I)\vec{x} = \vec{0}$.

$$\begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

② $\lambda = 3$. Solve $(A-3I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

4. $\underline{Q = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}} \Rightarrow Q^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix} = \underline{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = D}$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	35 (+2)	