國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） MATH 104 / GEAI 1209: Linear Algebra II

第二次期中考 May 6, 2019 **Midterm 2**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	8 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

– Enter your **Name** and **Student ID #** before you start.

– Using the calculator is not allowed (and not necessary) for this exam.

– Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.

– Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

– 可用中文或英文作答

1. [5pt] Let
$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$
Let $V = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a vector space. Use the Gram–Schmidt algorithm to find an **orthonormal** basis of V.

2. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 & 4 \\ 1 & 4 & 9 & 9 & 9 \\ 1 & 4 & 9 & 10 & 10 \\ 1 & 4 & 9 & 10 & 11 \end{bmatrix}.$$

Find $\det(\mathbf{A})$.

3. [2pt] Let

$$\mathbf{A} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}.$$

By the permutation expansion, $\det(\mathbf{A})$ is the sum of 24 terms. Find the signs of the following terms. [Write $+$ or $-$ in the blank.]

(a) ($\quad$)$bglm$

(b) ($\quad$)$chin$

(c) ($\quad$)$dejo$

(d) ($\quad$)$dgjm$

4. [1pt] Let $\mathbf{A}$ be a 5×5 matrix. According to the permutation expansion, how many terms are there in $\det(\mathbf{A})$? Also, how many terms in $\det(\mathbf{A})$ have negative signs?

5. [2pt] Let

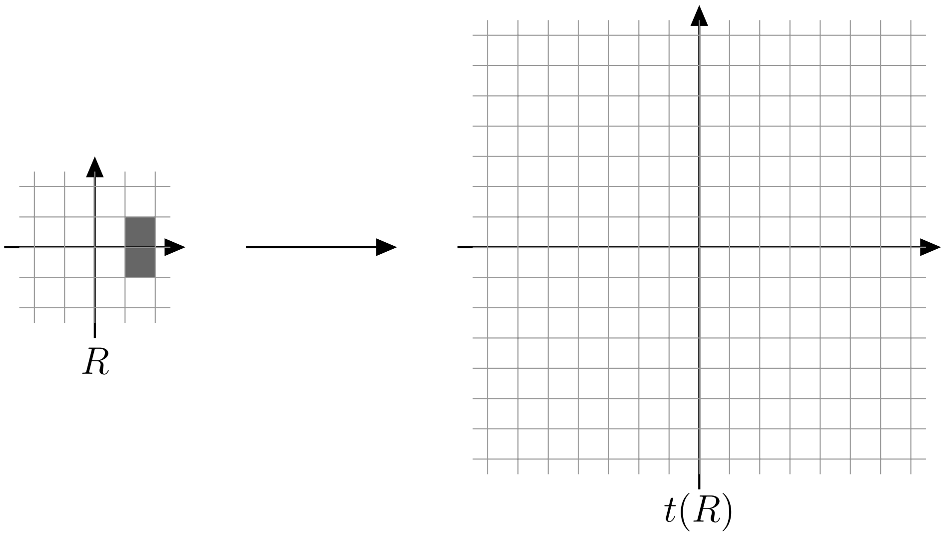
$$\mathbf{A} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ c & 0 & 0 & d \end{bmatrix}.$$

Find the formula of $\det(\mathbf{A})$.

6. [5pt] Let R be the rectangle enclosed by the four vertices

$$(1,1),(1,-1),(2,-1),(2,1).$$

Let $t:\mathbb{R}^2\to\mathbb{R}^2$ be a homomorphism defined by $t(\mathbf{x})=\mathbf{T}\mathbf{x}$ with $\mathbf{T}=\begin{bmatrix}1&1\\1&3\end{bmatrix}$. Draw the region $t(R)$ and compute its area.



7. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

(a) [1pt] Find $\det(\mathbf{A})$.

(b) [2pt] Let $\mathbf{A}^{-1}$ be the inverse of $\mathbf{A}$. Find the $4,3$-entry of $\mathbf{A}^{-1}$.

(c) [2pt] Find the $3,4$-entry of $\mathbf{A}^{-1}$.

8. [5pt] Let $\mathbf{A}$ be an $n \times n$ matrix whose columns are $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent, then $\det(\mathbf{A}) = 0$. [Prove by the definition of the determinant. Do not use the result that the determinant of singular matrix is zero.]

9. [5pt] Let $\mathbf{L}_n$ be the $n \times n$ matrix whose i, j-entry is -2 if $i = j$, 1 if $|i - j| = 1$, and 0 otherwise. For example,

$$\mathbf{L}_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \mathbf{L}_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \text{ and } \mathbf{L}_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Find the formula of $\det(\mathbf{L}_n)$ in terms of n. [You have to justify your answer.]

10. [extra 2pt] Let $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$. Follow the instructions below to find an invertible matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$.

    1. Compute the polynomial $p(t) = \det(\mathbf{A} - t\mathbf{I})$, where $\mathbf{I} = \mathbf{I}_2$ is the identity matrix.
    2. Solve $p(t) = 0$ and get the two roots $\Lambda = \{\lambda_1, \lambda_2\}$.
    3. For each $\lambda \in \Lambda$, compute a basis of the nullspace of $\mathbf{A} - \lambda\mathbf{I}$. (In this special case, say nullspace$(\mathbf{A} - \lambda_1\mathbf{I}) = \text{span}\{\mathbf{v}_1\}$ and nullspace$(\mathbf{A} - \lambda_2\mathbf{I}) = \text{span}\{\mathbf{v}_2\}$.)
    4. Let $\mathbf{Q}$ be the matrix whose columns are $\{\mathbf{v}_1, \mathbf{v}_2\}$. Then compute $\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$. (If everything works out, your $\mathbf{D}$ is a diagonal matrix.)

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	35 (+2)	