國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第二次期中考 May 6, 2019 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **8 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $V = \text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a vector space. Use the Gram–Schmidt algorithm to find an **orthonormal** basis of V.

$$\boxed{\operatorname{proj}_{[\vec{u}]}(\vec{v}) = \frac{\langle \vec{v}, \vec{u} \rangle}{|\vec{u}|^2} \cdot \vec{u}.}$$

Let $\vec{K}_1 = \vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{K}_2 = \vec{b}_2 - \operatorname{proj}_{[\vec{K}_1]}(\vec{b}_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{K}_3 = \vec{b}_3 - \operatorname{proj}_{[\vec{K}_1]}(\vec{b}_3) - \operatorname{proj}_{[\vec{K}_2]}(\vec{b}_3)$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1/2}{3/2}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}.$$

Thus, $\{\vec{K}_1, \vec{K}_2, \vec{K}_3\}$ is orthogonal.

$$\Rightarrow \underline{\left\{\frac{1}{\sqrt{2}}\vec{K}_1, \frac{1}{\sqrt{3/2}}\vec{K}_2, \frac{1}{\sqrt{4/3}}\vec{K}_3\right\}} \text{ is orthonormal.}$$

2. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 & 4 \\ 1 & 4 & 9 & 9 & 9 \\ 1 & 4 & 9 & 10 & 10 \\ 1 & 4 & 9 & 10 & 11 \end{bmatrix}.$$

Find $\det(\mathbf{A})$.

$$\det(A) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 3 & 3 & 3 & 3 \\ & 3 & 8 & 8 & 8 \\ & 3 & 8 & 9 & 9 \\ & 3 & 8 & 9 & 10 \end{pmatrix} = \det\begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 8 & 8 & 8 \\ 3 & 8 & 9 & 9 \\ 3 & 8 & 9 & 10 \end{pmatrix}$$

$$= \det\begin{pmatrix} 3 & 3 & 3 & 3 \\ & 5 & 5 & 5 \\ & 5 & \cancel{4}6 & \cancel{6}6 \\ & 5 & \cancel{4}6 & \cancel{5}7 \end{pmatrix} = \det\begin{pmatrix} 3 & 3 & 3 & 3 \\ & 5 & 5 & 5 \\ & & 1 & 1 \\ & & 1 & 2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 3 & 3 & 3 & 3 \\ & 5 & 5 & 5 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} = \underline{\underline{15}}.$$

3. [2pt] Let

$$\mathbf{A} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}.$$

By the permutation expansion, det(**A**) is the sum of 24 terms. Find the signs of the following terms. [Write + or − in the blank.]

(a) (−)$bglm$ $\longrightarrow \phi = 2\ 3\ 4\ 1$ 3 switches

(b) (+)$chin$ $\longrightarrow \phi = 3\ 4\ 1\ 2$ 2 switches

(c) (−)$dejo$ $\longrightarrow \phi = 4\ 1\ 2\ 3$ 3 switches

(d) (+)$dgjm$ $\longrightarrow \phi = \cancel{2\ 3\ 4\ 1}$

$4\ 3\ 2\ 1$ 2 switches

4. [1pt] Let **A** be a 5×5 matrix. According to the permutation expansion, how many terms are there in det(**A**)? Also, how many terms in det(**A**) has negative signs?

- det(A) has 5! = 120 terms
- 60 terms ~~are neg~~ have negative signs.

5. [2pt] Let

$$\mathbf{A} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ c & 0 & 0 & d \end{bmatrix}.$$

Find the formula of det(**A**).

$$\det(A) = a \cdot \det\begin{pmatrix} x & y & \\ z & w & \\ & & d \end{pmatrix} - b \cdot \det\begin{pmatrix} & x & y \\ & z & w \\ c & & \end{pmatrix}$$

$$= a \cdot d \cdot (xw - yz) - b \cdot (c) \cdot (xw - yz)$$

$$= (ad - bc)(xw - yz)$$

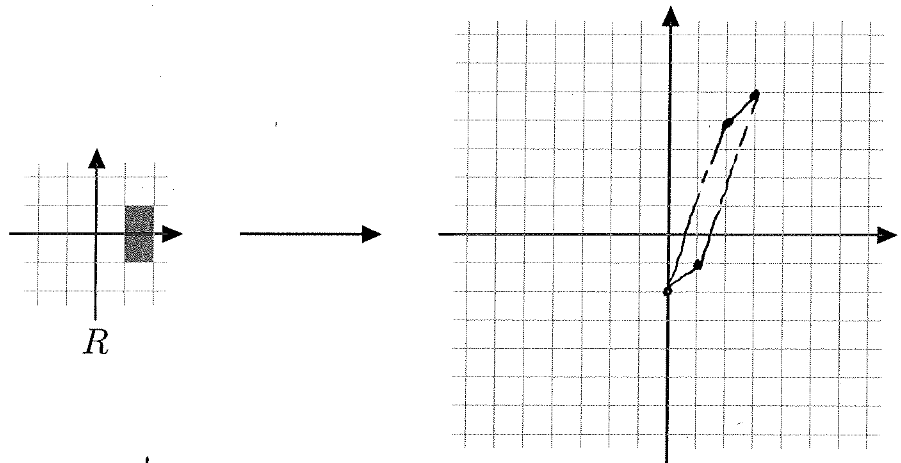
or

$$= adxw - adyz - bcxw + bcyz.$$

6. [5pt] Let R be the rectangle enclosed by the four vertices

$$(1, 1), (1, -1), (2, -1), (2, 1).$$

Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be a homomorphism defined by $t(\mathbf{x}) = \mathbf{T}\mathbf{x}$ with $\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. Draw the region $t(R)$ and compute its area.



R $t(R)$

$t(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$t(1,-1) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$t(2,-1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$t(2,1) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\text{Area} = \left| \det(T) \right| \cdot \text{area of } R$

$= 2 \cdot 2 = \underline{\underline{4}}.$

7. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

(a) [1pt] Find $\det(\mathbf{A})$.

$\det(A) = 5! = 120$

(b) [2pt] Let $\mathbf{A}^{-1}$ be the inverse of $\mathbf{A}$. Find the $4,3$-entry of $\mathbf{A}^{-1}$.

$4,3$-entry of $A^{-1} = [4,3\text{-entry of } \operatorname{adj}(A)] / \det(A)$

$= [3,4\text{-cofactor of } A] / \det(A)$

$= (-1)^7 \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 2 & 1 & 1 \\ & & 0 & 1 \\ & & & 5 \end{pmatrix} / 120$

$= 0.$

(c) [2pt] Find the $3,4$-entry of $\mathbf{A}^{-1}$.

$3,4$-entry of $A^{-1} = [4,3\text{-cofactor of } A] / \det(A)$

$= (-1)^7 \cdot \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 2 & 1 & 1 \\ & & 1 & 1 \\ & & & 5 \end{pmatrix} / 120$

$= -\dfrac{10}{120} = -\dfrac{1}{12}.$

8. [5pt] Let $\mathbf{A}$ be an $n \times n$ matrix whose columns are $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent, then $\det(\mathbf{A}) = 0$.

See ver. A.

9. [5pt] Let $\mathbf{L}_n$ be the $n \times n$ matrix whose i,j-entry is -2 if $i = j$, 1 if $|i-j| = 1$, and 0 otherwise. For example,

$$\mathbf{L}_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \mathbf{L}_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \text{ and } \mathbf{L}_4 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Find the formula of $\det(\mathbf{L}_n)$ in terms of n. [You have to justify your answer.]

See ver. A.

10. [extra 2pt] Let $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$. Follow the instructions below to find an invertible matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$.

1. Compute the polynomial $p(t) = \det(\mathbf{A} - t\mathbf{I})$, where $\mathbf{I} = \mathbf{I}_2$ is the identity matrix.
2. Solve $p(t) = 0$ and get two roots $\Lambda = \{\lambda_1, \lambda_2\}$.
3. For each $\lambda \in \Lambda$, compute a basis of the nullspace of $\mathbf{A} - \lambda\mathbf{I}$. (In this special case, say nullspace$(A - \lambda_1\mathbf{I}) = \text{span}\{\mathbf{v}_1\}$ and nullspace$(\mathbf{A} - \lambda_2\mathbf{I}) = \text{span}\{\mathbf{v}_2\}$.)
4. Let $\mathbf{Q}$ be the matrix whose columns are $\{\mathbf{v}_1, \mathbf{v}_2\}$. Then compute $\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$. (If everything works out, your $\mathbf{D}$ is a diagonal matrix.)

See. Ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	35 (+2)	