Sample Solutions for Sample Questions 10.

1. ~~8~~

Since $\underbrace{A}_{\text{矩陣}} \cdot \underbrace{\mathrm{adj}(A)}_{\text{矩陣}} = \underbrace{\det(A)}_{\text{純量}} \cdot \underbrace{I_n}_{\text{單位矩陣}}$.

$$\det(A) \cdot \det\left(\mathrm{adj}(A)\right) = \det(A)^n$$

$$\Rightarrow \quad \det\left(\mathrm{adj}(A)\right) = \det(A)^{n-1} = 2^{n-1}.$$

2. By permutation expansion (or Laplace expansion),

~~det~~ $\det A = 2abcde$.

Therefore, when $a, b, c, d, e \neq 0$,

$\det(A) \neq 0$, so A is invertible.

3.

$$\det A(t) = \det\begin{pmatrix} t & 1 & & & \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & & & 1 & \end{pmatrix} = \det\begin{pmatrix} t & 1 & & & \\ 0 & & 1 & & \\ 0 & 1 & & 1 & \\ 0 & & 1 & & 1 \\ 0 & & & 1 & \end{pmatrix} + \det\begin{pmatrix} & 1 & & & \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & & & 1 & \end{pmatrix}$$

$$= t \cdot \underbrace{\det\begin{pmatrix} & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{pmatrix}}_{\substack{4\times 4 \\ \uparrow \\ a}} + \underbrace{\det\begin{pmatrix} & 1 & & & \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & & & 1 & \end{pmatrix}}_{\substack{5\times 5 \\ \uparrow \\ b}}$$

So $\quad a = \det\begin{pmatrix} & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{pmatrix} = 1.$

4.

$A(t)\cdot\vec{x}=\vec{0}$ 有非零解 $\Leftrightarrow$ $A(t)$ is singular

$\Leftrightarrow$ $\det A(t) = 0$.

Compute $\det A(t) = (1-t)^3 + 1 + 1 - (1-t) - (1-t) - (1-t)$

$$= (1-t)^3 - 3(1-t) + 2$$

$$\cancel{= (1-t)[(1-t)}$$

$$= (-t^3+3t^2-3t+1) - (3-3t) + 2$$

$$= -t^3+3t^2 = -t^2(t-3)$$

$\Rightarrow$ so $\det(A(t)) = 0$ when $t = \underbrace{0,0}_{重根}, 3$.

① $t=0$.

$$A(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\begin{array}{ccc} & \uparrow & \uparrow \\ & \text{free} & \end{array}$$

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

② $t=3$

$$A(3) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ \\ \leftarrow \text{free} \end{array}$$

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

5. Same process as in Problem 4.

$$\det A(t) = (2-t)(1-t)^2 - (1-t) - (1-t)$$

$$= (2-t)(1-t)^2 - 2(1-t)$$

$$= (1-t)\left[(2-t)(1-t) - 2\right]$$

$$= (1-t)(2-3t+t^2-2)$$

$$= (1-t)(t^2-3t) = t(1-t)(t-3)$$

$\Rightarrow$ $\det A(t) = 0$ when $t = 0, 1, 3$.

① $t=0$, $A(0) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \\ -1 & & 1 \end{pmatrix} \to \begin{pmatrix} -1 & & 1 \\ -1 & 1 & \\ 2 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & & 1 \\ & 1 & -1 \\ & & \end{pmatrix}$

free.

$\Rightarrow$ nullspace $= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

② $t=1$, $A(1) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & \\ -1 & & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ & -1 & -1 \\ & & \end{pmatrix}$

free

$\Rightarrow$ nullspace $= \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$.

③ $t=3$ $A(3) = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2 & \\ -1 & & -2 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & -1 \\ & -1 & 1 \\ & & \end{pmatrix}$

free.

$\Rightarrow$ nullspace $= \operatorname{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$

6, 7. 你可以去算 det A(t)，但很花時間.

或是用電腦算（但考試沒電腦）.

Here are some doable methods.

6.

$$\det A(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 .$$

$$\Rightarrow \quad a_0 = \det A(0) = \det \begin{pmatrix} & 1 & & & \\ 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ & & & 1 & \end{pmatrix} = 0 .$$

$b_{i,j}$ 是 $A(t)$ 的 i,j-entry

$$\det A(t) = \sum_{\phi:\ 5\text{-permutation}} \underbrace{b_{1,\phi(1)} \cdots b_{5,\phi(5)}}_{\uparrow} \cdot \underbrace{\det P_\phi}_{\pm 1} .$$

degree of t = # of fixed points

（有幾個 k 使得 $\phi(k)=k$）

$$\text{So} \quad a_5 t^5 = \sum_{\substack{\phi:\ 5\text{-perm} \\ \phi \text{ has exactly} \\ 5 \text{ fixed points}}} b_{1,\phi(1)} \cdots b_{5,\phi(5)} \cdot \det P_\phi .$$

只有 $\phi = id$ 才有 5 個 fixed point.

$$\Rightarrow a_5 t^5 = \underbrace{b_{1,1}}_{-t} \cdots\cdots \underbrace{b_{5,5}}_{-t} \cdot \underbrace{\det P_{id}}_{\text{單位矩陣}}$$

$$= (-t)^5 = -t^5 .$$

$$\Rightarrow a_5 = -1 .$$

7.

Let $u_1, u_2, \ldots, u_5$ be the columns of $\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} = A(0)$

Let $v_1, \ldots, v_5$ be the columns of $\begin{pmatrix} -t & & & & \\ & -t & & & \\ & & -t & & \\ & & & -t & \\ & & & & -t \end{pmatrix}$

So $\det A(t) = \det \begin{pmatrix} | & & | \\ u_1+v_1 & \cdots & u_5+v_5 \\ | & & | \end{pmatrix} = \underbrace{\text{sum of } 2^5 \text{ determinant}}_{}$.

每個column可拆成u或v.

e.g. 其中一項是 $\begin{pmatrix} | & | & | & | & | \\ u_1 & v_2 & u_3 & v_4 & u_5 \\ | & | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & & 0 & & 0 \\ & -t & & & \\ 0 & & 0 & & 0 \\ & & & -t & \\ 0 & & 0 & & 0 \end{pmatrix}$

$\Rightarrow \det = (-t)^2 \cdot \det \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

扣掉 2,4 row/column

Let $B = A(0)$.

Define $B(\alpha) = B$ remove rows/colums in α.

e.g. $B(\{1,2,3\}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B(\{1,2,3,4\}) = (0)$.

Thus, $a_1 t = \sum_{\substack{\alpha \subseteq \{1,2,3,4,5\} \\ |\alpha| = 1}} (-t)^1 \cdot B(\alpha) = (1 + 0 + 1 + 0 + 1)(-t)$

$\underbrace{\det\begin{pmatrix} & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{pmatrix}}_{B(\{1\})}$ $\underbrace{\det\begin{pmatrix} 0 & & & \\ & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{pmatrix}}_{B(\{2\})}$ $\cdots$

$\Rightarrow a_1 = -3$.

Similarly, $a_4 t^4 = \sum_{\substack{\alpha \subseteq \{1,2,3,4,5\} \\ |\alpha| = 4}} (-t)^4 B(\alpha) = t^4 \cdot (0+0+0+0+0)$.

$\Rightarrow a_4 = 0$.