Sample Questions 11

Let $\mathcal{P}_n$ be the space of all polynomials with real coefficients and with degree at most n.

1. Let $V = \mathcal{P}_1 \oplus \mathcal{P}_2$ be the direct sum of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then every element in V is of the form $(a(x), b(x))$ with $a(x) \in \mathcal{P}_1$ and $b(x) \in \mathcal{P}_2$. Compute
$$(1+2x, 3+2x+x^2) + (1-x, 1-x+x^2)$$
and
$$5(1+2x, 3+2x+x^2).$$
Also, find a basis of V.

2. Let $V = \mathcal{P}_1 \times \mathcal{P}_2$. Let $f : V \to \mathcal{P}_4$ be a mapping defined by
$$f(a(x), b(x)) = a(x)p(x) + b(x)q(x),$$
where $p(x) = 4 + 8x + 5x^2 + x^3$ and $q(x) = -1 + x^2$. Let
$$\mathcal{B} = \{(1,0), (x,0), (0,1), (0,x), (0,x^2)\}$$
be a basis of V and let
$$\mathcal{D} = \{1, x, x^2, x^3, x^4\}$$
be a basis of $\mathcal{P}_4$. Show that f is a homomorphism and find the matrix representation $\text{Rep}_{\mathcal{B},\mathcal{D}}(f)$. Can you find the nullspace fo f?

3. Let $p(x) = 1 + 2x + x^2$ and $q(x) = 2 + 3x + x^2$. Find the Sylvester matrix $S_{p,q}$ of $p(x)$ and $q(x)$. Also, find the resultant $\text{Res}(p, q)$ of $p(x)$ and $q(x)$. Based on this, do $p(x)$ and $q(x)$ have a common root?

4. Diagonalize $\mathbf{A} = \begin{bmatrix} -2 & 15 \\ 1 & 0 \end{bmatrix}$.

5. Diagonalize $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

6. Diagonalize $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

7. Diagonalize $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.