Sample Solution for Sample Questions 11.

1. $(1+2x,\ 3+2x+x^2)+(1-x,\ 1-x+x^2)$

$=(1+2x+1-x,\ 3+2x+x^2+1-x+x^2)$

$=(2+x,\ 4+x+2x^2).$

$5(1+2x,\ 3+2x+x^2)$

$=(5+10x,\ 15+10x+5x^2)$

A basis can be

$\{(1,0),(x,0),(0,1),(0,x),(0,x^2)\}.$

2. $$f: \underset{\mathcal{P}_1 \oplus \mathcal{P}_2}{\overset{\|}{V}} \longrightarrow \mathcal{P}_4$$

$$(a(x), b(x)) \longmapsto a(x)\cdot p(x) + b(x)\cdot q(x).$$

$p(x) = 4+8x+5x^2+x^3$　　$q(x) = -1+x^2$　給定的.

① 寫在後面

② Compute

$$f(1,0) = p(x) = 4+8x+5x^2+x^3 \xrightarrow{Rep_D} \begin{pmatrix} 4 \\ 8 \\ 5 \\ 1 \\ 0 \end{pmatrix}$$

$$f(x,0) = x\cdot p(x) = 4x+8x^2+5x^3+x^4 \xrightarrow{Rep_D} \begin{pmatrix} 0 \\ 4 \\ 8 \\ 5 \\ 1 \end{pmatrix}$$

$$f(0,1) = q(x) = -1+x^2 \xrightarrow{Rep_D} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$f(0,x) = x\,q(x) = -x+x^3 \xrightarrow{Rep_D} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f(0,x^2) = x^2 q(x) = -x^2+x^4 \xrightarrow{Rep_D} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow Rep_{B,D}(f) = \begin{pmatrix} 4 & 0 & -1 & 0 & 0 \\ 8 & 4 & 0 & -1 & 0 \\ 5 & 8 & 1 & 0 & -1 \\ 1 & 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

2 [continued].

① To show that f is a homomorphism.

~~check~~ $\cancel{f((a_1,b_1)+(a_2,b_2)) = f((a_1+a_2,\ b_1+b_2))}$

$\cancel{f(k(a,b)) = f(}$

check $f((a_1,b_1)+(a_2,b_2)) = f(a_1,b_1) + f(a_2,b_2)$

and $f(k(a,b)) = k \cdot f(a,b)$.

(i)

$$f((a_1,b_1)+(a_2,b_2)) = f((a_1+a_2,\ b_1+b_2))$$

$$= (a_1+a_2)p + (b_1+b_2)q$$

$$= (a_1p+b_1q) + (a_2p+b_2q)$$

$$= f(a_1,b_1) + f(a_2,b_2).$$

(ii) $f(k(a,b)) = f(ka, kb)$

$$= kap + kbq$$

$$= k(ap+bq)$$

$$= k \cdot f(a,b).$$

So f is a homomorphism.

2 [continued].

③. To find the nullspace,
you may compute nullspace($Rep_{B,D}(f)$)
and the find the nullspace of f.

Alternatively, solve $a(x)p(x) + b(x)q(x) = 0$
with $\deg a \leq 1$, $\deg b \leq 2$.

Note $p(x) = 4 + 8x + 5x^2 + x^3$

$= (x+1)(x+2)^2$

$q(x) = -1 + x^2$

$= (x+1)(x-1)$.

$\Rightarrow \quad a(x)(x+1)(x+2)^2 = -b(x)\cdot(x+1)(x-1)$

$\Rightarrow \quad a(x)$ has a factor $x-1$

and $b(x) = -\dfrac{a(x)}{x-1}\cdot(x+2)^2$

Equivalently, write $a(x) = k(x-1)$

$\Rightarrow \quad b(x) = -k(x+2)^2$.

So nullspace $(f) = \{(k(x-1), -k(x+2)^2) : k \in R\}$

$= \text{span}\{\cancel{(x+1)}(x-1, -(x+2)^2)\}$.

3. $p(x)$ has deg 2
$q(x)$ has deg 2 $\Rightarrow$ Sylvester matrix is $(2+2) \times (2+2)$.

$$\Rightarrow S_{p,q}(x) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$Res(p,q) = \det(S_{p,q}(x))$$

$$= \det \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = 0.$$

So $p(x)$ and $q(x)$ have a common root.

(of course, in the small example,

$$p(x) = 1+2x+x^2 = (x+1)^2$$

$$q(x) = 2+3x+x^2 = (x+1)(x+2)$$

have a common root.)

4. $A = \begin{pmatrix} -2 & 15 \\ 1 & 0 \end{pmatrix}$.

① Find eigenvalues.

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 15 \\ 1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 + 2\lambda - 15 = (\lambda - 3)(\lambda + 5).$$

$\Rightarrow \lambda = 3, -5.$

② Find eigenvectors for each eigenvalue.

(i) $\lambda = 3$. $A - 3I = \begin{pmatrix} -5 & 15 \\ 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.

(ii). $\lambda = -5$, $A + 5I = \begin{pmatrix} 3 & 15 \\ 1 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 5 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} 5 \\ -1 \end{pmatrix} \right\}$.

Thus, $AQ = QD$ with $Q = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} -3 & \\ & 5 \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$.

5. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

① Find eigenvalues.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

$$= (\lambda + i)(\lambda - i).$$

$\Rightarrow \lambda = \pm i.$

[這個A無法在ℝ中對角化].

② Find eigenvector for each eigenvalue.

$\lambda = i$. $A - iI = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$.

$\Rightarrow$ ~~nullspace~~ nullspace $= \text{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$

$\lambda = -i$. $A + iI = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

Thus, $AQ = QD$ with $Q = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} i & \\ & -i \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$.

6.

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \\ -1 & & 1 \end{pmatrix}$$

① Find eigenvalues

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 1-\lambda & \\ -1 & & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2 - 3\lambda + 2 \quad -2\right]$$

$$= (1-\lambda)(\lambda - 3)\cdot\lambda \qquad \Rightarrow \lambda = 0, 1, 3.$$

② Find eigenvectors for each eigenvalue.

(i) $\lambda = 0$. $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \\ -1 & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & \\ & -1 & 1 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

(ii) $\lambda = 1$ $\quad A - I = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & \\ -1 & & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$.

(iii) $\lambda = 3$, $\quad A - 3I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2 & \\ -1 & & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ & 1 & -1 \end{pmatrix}$

$\Rightarrow$ null space $= \text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$.

Thus, $AQ = QD$ with $Q = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 3 \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$.

7. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

① Find eigenvalues.

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} =$$

$$= -(\lambda+1)^2 \cdot (\lambda-2). \quad \Rightarrow \lambda = -1, -1, 2.$$

② Find eigenvector for each eigenvalue.

(i) $\lambda = -1$. $\quad A+I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

$\Rightarrow$ nullspace $= \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$\lambda=-1$ 有 2個重根，eigenvector "通常" 有兩個.

(ii) $\lambda = 2$. $\quad A-2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & \end{pmatrix}$

$\Rightarrow$ ~~null~~ nullspace $= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$

Thus, $AQ = QD$ with $Q = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$