Sample Solutions for Sample Questions 12.

1. Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{D} = \{\vec{u}_1, \ldots, \vec{u}_m\}$.

Recall that

$$\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f) = \begin{pmatrix} | & & | \\ \mathrm{Rep}_{\mathcal{D}} f(\vec{v}_1) & \cdots & \mathrm{Rep}_{\mathcal{D}} f(\vec{v}_n) \\ | & & | \end{pmatrix}$$

$$\shortparallel$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \\ 0 & 1 & & \vdots & \\ \vdots & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 0 & \end{pmatrix}$$

This means

$$f(\vec{v}_1) = \vec{u}_1, \ \ldots \ f(\vec{v}_r) = u_r \quad \text{with } r = \mathrm{rank}(f).$$

also $f(\vec{v}_{r+1}) = \cdots = f(\vec{v}_n) = \vec{0}$.

Recall that $\mathrm{nullity}(f) + \mathrm{rank}(f) = n \Rightarrow \mathrm{nullity}(f) = n - r$.

Naturally, pick $\{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$ as a basis of ~~nullspace(f)~~ nullspace(f).

Also, pick $\{\vec{v}_1, \ldots, \vec{v}_r\}$ as a basis of $\mathrm{nullspace}(f)^{\perp}$

$\dim = n-(n-r) = r.$

Next, let $\vec{u}_i = f(\vec{v}_i)$ for $i = 1 \sim r$.

$\{\vec{u}_1, \ldots, \vec{u}_r\}$ form an ~~span~~ independent set and span a subspace U in W.

~~Take~~ Expand $\{\vec{u}_1, \ldots, \vec{u}_r\}$ into a basis $\{\vec{u}_1, \ldots, \vec{u}_m\}$ of W.

Then we are done.

2. Compute the char poly $p(x)$.

$$\det\begin{pmatrix} -x & 1 & 1 & 1 \\ 1 & -x & 1 & 1 \\ 1 & 1 & -x & 1 \\ 1 & 1 & 1 & -x \end{pmatrix} = -x\cdot\underbrace{\det\begin{pmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{pmatrix}}_{-x^3+2+3x} - \underbrace{\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{pmatrix}}_{q(x)=x^2+1+2x}$$

$$+\underbrace{\det\begin{pmatrix} 1 & -x & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -x \end{pmatrix}}_{-q(x)} - \underbrace{\det\begin{pmatrix} 1 & -x & 1 \\ 1 & 1 & -x \\ 1 & 1 & 1 \end{pmatrix}}_{q(x)}$$

$$= -x(-x^3+3x+2) - 3q(x)$$

$$= -x(-x^3+3x+2) - 3(x^2+2x+1)$$

$$= x(x+1)\underbrace{(x^2-x-2)}_{(x+1)(x-2)} - 3(x+1)^2$$

$$= (x+1)^2[x^2-2x-3] = (x+1)^3(x-3)$$

$$\begin{array}{cccc} -1 & 0 & 3 & 2 \\ -1 & & 1 & 2 \\ & 1 & 1 & \end{array}$$

~~eigenvalues = 1, 1, 1, 3.~~

$\Rightarrow \lambda = -1, -1, -1, 3.$

① $\lambda = -1 \Rightarrow A+I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_{-1} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. (geo multi = 3, good!)

② $\lambda = 3$, $A - 3I = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & -8 & 4 & 4 \\ 1 & -3 & 1 & 1 \\ 0 & 4 & -4 & 0 \\ 0 & 4 & 0 & -4 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 1 & -3 & 1 & 1 \\ & 1 & -1 & \\ & & 1 & -1 \end{pmatrix}$

$E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

Let $D = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 3 \end{pmatrix}$, $Q = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Then $Q^{-1}AQ = D$.

3. Compute char poly $p(x)$:

$$\det\begin{pmatrix} -x & 1 & & 1 \\ 1 & -x & 1 & \\ & 1 & -x & 1 \\ 1 & & 1 & -x \end{pmatrix} = -x\det\underbrace{\begin{pmatrix} -x & 1 & 0 \\ 1 & -x & 1 \\ 0 & 1 & -x \end{pmatrix}}_{-x^3+2x} - \det\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & -x & 1 \\ 1 & 1 & -x \end{pmatrix}}_{q(x) = x^2}$$

$$-\det\underbrace{\begin{pmatrix} 1 & -x & 1 \\ 0 & 1 & -x \\ 1 & 0 & 1 \end{pmatrix}}_{x^2}$$

$$= -x(-x^3+2x) - 2x^2$$

$$= x^4 - 2x^2 - 2x^2 = x^4 - 4x^2 = x^2(x+2)(x-2).$$

$\Rightarrow \lambda = 0, 0, \pm 2.$

① $\lambda = 0$. $A - 0I = \begin{pmatrix} & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$\Rightarrow E_0 = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

② $\lambda = 2$, $A - 2I = \begin{pmatrix} -2 & 1 & & 1 \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ 1 & & 1 & -2 \end{pmatrix} \Rightarrow E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$

③ $\lambda = -2$, $A + 2I = \begin{pmatrix} 2 & 1 & & 1 \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ 1 & & 1 & 2 \end{pmatrix} \Rightarrow E_{-2} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right\}.$

Let $D = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & -2 \end{pmatrix}$, $Q = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

Then $Q^{-1}AQ = D$.

4. Char poly $p(x) = \det(A - xI)$

$$= \det\begin{pmatrix} 2-x & & 1 & 1 \\ & 2-x & 1 & 1 \\ & & 1-x & \\ & & & 1-x \end{pmatrix} = (x-2)^2(x-1)^2.$$

$\Rightarrow \lambda = 1, 1, 2, 2.$

① $\lambda = 1.$ $A - I = \begin{pmatrix} 1 & & 1 & 1 \\ & 1 & 1 & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$\Rightarrow E_1 = \operatorname{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

② $\lambda = 2,$ $A - 2I = \begin{pmatrix} 0 & & 1 & 1 \\ & 0 & 1 & 1 \\ & & -1 & \\ & & & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow E_2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}.$

5.

char poly $p(x) = \det(A - xI)$

$$= \det\begin{pmatrix} 2-x & 1 & 1 & 1 \\ & 2-x & 1 & 1 \\ & & 1-x & 1 \\ & & & 1-x \end{pmatrix} = (x-2)^2(x-1)^2.$$

$\Rightarrow \lambda = 1, 1, 2, 2.$

① $\lambda = 1$ $\quad A - I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

$\Rightarrow$ $\dim E_1 = 1$ $\quad$ only one free variable.

$1 =$ geo multi $\neq$ alg multi $= 2$.

So A is not diagonalizable.

6. Let $A = \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & 2 & 0 & 1 \\ & & & 2 & 0 \\ & & & & 2 \end{pmatrix}$

$\Rightarrow$ char poly $= (x-1)^2(x-2)^3$

$\Rightarrow \begin{cases} \lambda = 1 \text{ has alg multi} = 2 \\ \lambda = 2 \text{ has alg multi} = 3. \end{cases}$

$A - I$ has 1 free variable $\Rightarrow \dim E_1 = 1$

$A - 2$ has 2 free variables $\Rightarrow \dim E_2 = 2$

7. Instead of find the eigenvalue 1,

we find an ~~gen~~ eigenvector $\vec{x}$ with $x^T M = x^T$.

Let $1^T = [1 \ 1 \ \cdots \ 1]$.

By the defn of a stochastic matrix,

any such matrix M has $1^T M = 1^T$.

So $1^T (M - I) = 0^T$

$\Rightarrow$ $M - I$ is not invertible

$\Rightarrow$ $\det(M - I) = 0$.

$\Rightarrow$ 1 is an eigenvalue.