

Sample Solutions for Sample Questions 13 ~~Math~~

1. ~~Check~~

Since t is the orthogonal projection onto V ,

$$t(\vec{v}_1) = \vec{v}_1$$

$$t(\vec{v}_2) = \vec{v}_2$$

$$t(\vec{v}_3) = 0 \cdot \vec{v}_3$$

Let $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Then $\text{Rep}_{B,B}^{(t)} = \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}}_T$

\Rightarrow eigenvalues = 1, 1, 0.

eigenspaces:

$$E_1 = \{ \vec{v} \mid t(\vec{v}) = 1 \cdot \vec{v} \}$$

$$= \text{span} \{ \vec{v}_1, \vec{v}_2 \}$$

$$E_0 = \{ \vec{v} \mid t(\vec{v}) = 0 \cdot \vec{v} \}$$

$$= \text{span} \{ \vec{v}_3 \}$$

characteristic polynomial

$$\phi(x) = \det(T - xI) = (1-x)(1-x)(0-x)$$

2.

$$\text{Reflection: } \begin{cases} t(\vec{v}_1) = \vec{v}_1 \\ t(\vec{v}_2) = \vec{v}_2 \\ t(\vec{v}_3) = -\vec{v}_3 \end{cases}$$

Let $B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ be an orthonormal basis of \mathbb{R}^3 .

$$\Rightarrow \text{Rep}_{B-B}^t = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

3. [這題跟筆記一樣, 練習寫一下]

Suppose A and B are similar.

$\Rightarrow \exists Q$ (Q invertible) such that $Q^{-1}AQ = B$.

Since $\det(Q^{-1}) \cdot \det(Q) = \det(I) = 1$,

char poly of $A = \det(A - xI)$

$$= \det(Q^{-1}) \det(A - xI) \det(Q)$$

$$= \det(Q^{-1}(A - xI)Q)$$

$$= \det(Q^{-1}AQ - xQ^{-1}IQ)$$

$$= \det(B - xI) = \text{char poly of } B.$$

4. [This example show you that you do not always have to use the char poly to find the eigenvalues.]

① Observe that

$$J_n \cdot \mathbf{1}_n = n \cdot \mathbf{1}_n$$

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix}$$

$\Rightarrow n$ is an eigenvalue of geo mult at least 1.

② Observe that $\text{rank } J_n = 1$.

$$\Rightarrow \dim \text{nullspace}(J_n) = n-1$$

$$\Rightarrow \dim E_0 = n-1$$

\uparrow
eigenspace with respect to 0.

Now eigenvalues	0	n
geo mult	$n-1$	≥ 1
alg mult	$\geq n-1$	≥ 1

But $p(x) = \det(J_n - xI)$ is a poly of degree n .

$$\Rightarrow p(x) = (-1)^n (x-0)^{n-1} (x-n)^1 = (-1)^n \cdot x^{n-1} (x-n)$$

\uparrow
the leading coefficient of $\det(A - xI)$
for an $n \times n$ matrix A
is always $(-1)^n$.

5.

⊙ Compute an eigenvalue.

$$\det(A - xI) = \det \begin{pmatrix} -x & 1 & 1 \\ -1 & 2-x & 2 \\ 0 & 0 & 1-x \end{pmatrix} = (x^2 - 2x + 1)(1-x) = -(x-1)^3$$

Let $\lambda = 1$. Find an eigenvector.

$$A - I = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

↑
free.

$$E_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{Pick a unit eigenvector } \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Expand $\{\vec{v}_1\}$ to an orthonormal basis

$$\left\{ \vec{v}_1, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

可以用 Gram-Schmidt
或用湊的。

$$Q^{(1)} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{(1)} = [Q^{(1)}]^* T Q^{(1)} = \begin{pmatrix} 1 & -2 & 3/\sqrt{2} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

(Here $T = A$.)

因為 $T \vec{v}_1 = \vec{v}_1$

$$T \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -\frac{2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

也可以直接計算

[5 continued]

$$\text{Let } Q = Q^{(1)} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $Q^T A Q$ is upper triangular.

6. Suppose $U = [u_{i,j}]$.

U 矩陣的 ij -entry 叫 $u_{i,j}$.

Since U is upper triangular,

$$u_{i,j} = 0 \text{ if } i > j. \quad \dots (1)$$

Compare the 1,1-entry of UU^T and U^*U .

$$\Rightarrow \sum_{k=1}^n u_{1,k} \bar{u}_{1,k} = \bar{u}_{1,1} \cdot u_{1,1}$$

$$\left(\begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \right) = \left(\begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \right)$$

$$\Rightarrow |u_{1,1}|^2 + |u_{1,2}|^2 + \dots + |u_{1,k}|^2 = |u_{1,1}|^2$$

$$\Rightarrow u_{1,2} = u_{1,3} = \dots = u_{1,k} = 0.$$

Next compare the 2,2-entry, the 3,3-entry, ...

$\Rightarrow U$ is diagonal.

7. Do diagonalization on your own.

$$\lambda = 0 \Rightarrow E_0 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \quad \text{Pick } \vec{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \Rightarrow E_1 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}. \quad \text{Pick } \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda = 3 \Rightarrow E_3 = \text{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}. \quad \text{Pick } \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Let } Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 3 \end{pmatrix}$$

$$\Rightarrow Q^T A Q = D$$

$$\Rightarrow A = Q D Q^T$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 1 & \\ & & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= 0 \cdot \vec{v}_1 \vec{v}_1^T + 1 \cdot \vec{v}_2 \vec{v}_2^T + 3 \cdot \vec{v}_3 \vec{v}_3^T$$

$$= 0 \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + 1 \cdot \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + 3 \cdot \frac{1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$