Sample Solution for Sample Question 14.

1. To show that $\left(\frac{1}{\sqrt{n}}Q^*\right)\left(\frac{1}{\sqrt{n}}Q\right) = I$,

we show $Q^*Q = nI$ instead.

The j,k-entry of Q^*Q is

$$\sum_{l=1}^{n} (Q^*)_{j,l}\,(Q)_{l,k} = \sum_{l=1}^{n} \overline{w^{(l-1)(j-1)}} \cdot w^{(l-1)(k-1)}$$

$$= \sum_{l=1}^{n} w^{-(l-1)(j-1)+(l-1)(k-1)} = \sum_{l=1}^{n} w^{(k-j)(l-1)}$$

$$= \sum_{l=1}^{n} r^{l-1}, \quad \text{where } r = w^{k-j}.$$

$$= \begin{cases} n & \text{if } r = 1 \\ \dfrac{r^n - 1}{r-1} & \text{if } r \neq 1 \end{cases}$$

$$= \begin{cases} n & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \quad \left(\text{since } r^n = w^{(k-j)\cdot n} = (w^n)^{k-j} = 1.\right)$$

2. $A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.

$S_0 = 1$

$S_1 = 3+4 = 7$

$S_2 = 3\cdot 4 - 5\cdot 2 = 2$

$\Rightarrow p(x) = S_0(-x)^2 + S_1(-x)^1 + S_2(-x)^0$

$= x^2 - 7x + 2$.

3. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$S_0 = 1$

$S_1 = 1+2+9 = 12$

$S_2 = 1+8+6 = 15$

$S_3 = \det(A) = 2$

α	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$A[\alpha]$	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 9 \end{pmatrix}$	$\begin{pmatrix} 2 & 4 \\ 3 & 9 \end{pmatrix}$
det	1	8	6

$\Rightarrow p(x) = S_0(-x)^3 + S_1(-x)^2 + S_2(-x)^1 + S_3(-x)^0$

$= -x^3 + 12x^2 - 15x + 2$.

4. Observe that

$$\det(J_k) = \begin{cases} 0 & \text{if } k \geq 2 \\ 1 & \text{if } k = 1. \end{cases}$$

$$S_0 = 1$$

$$S_1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ 個}} = n$$

$$S_2 = \cdots = S_n = 0.$$

$$\Rightarrow p(x) = S_0(-x)^n + S_1(-x)^{n-1} + \cdots + S_n(-x)^0$$

$$= (-1)^n x^n + (-1)^{n-1} \cdot n \cdot x^{n-1}$$

$$= (-1)^n x^{n-1} \cdot (x - n).$$

5. Recall that $\det(J_n - xI_n) = (-1)^n x^{n-1} \cdot (x-n)$.

Thus,

$$\det(J_n - I_n - xI_n) = \det(J_n - (x+1) I_n)$$

$$= (-1)^n (x+1)^{n-1} (x+1-n).$$

用 $x+1$ 代入 x.

6. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

Then

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_0 I$$

$$\begin{aligned} p(Q^{-1}AQ) &= a_n (Q^{-1}AQ)^n + a_{n-1} (Q^{-1}AQ)^{n-1} + \dots + a_0 I \\ &= a_n Q^{-1} A^n Q + a_{n-1} Q^{-1} A^{n-1} Q + \dots + a_0 I \\ &= Q^{-1} (a_n A^n Q + a_{n-1} A^{n-1} Q + \dots + a_0 Q) \\ &= Q^{-1} (a_n A^n + a_{n-1} A^{n-1} + \dots + a_0) Q \\ &= Q^{-1} p(A) Q . \end{aligned}$$

7. Let A be a diagonalizable

and ~~supp~~ assume $Q^{-1}AQ = D$,

where D is a diagonal matrix.

Since $A \underset{\text{similar}}{\sim} D$, they have the same characteristic polynomial $p(x)$.

We may assume $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

and $p(x) = (-1)^n (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

Check that

$p(D) = (-1)^n (D-\lambda_1 I)(D-\lambda_2 I)\cdots(D-\lambda_n I)$.

$$= (-1)^n \begin{pmatrix} 0 & & & \\ & \lambda_2-\lambda_1 & & \\ & & \lambda_3-\lambda_1 & \\ & & & \ddots \\ & & & & \lambda_n-\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1-\lambda_2 & & & \\ & 0 & & \\ & & \lambda_3-\lambda_2 & \\ & & & \ddots \\ & & & & \lambda_n-\lambda_2 \end{pmatrix} \cdots \begin{pmatrix} \lambda_1-\lambda_n & & & \\ & \lambda_2-\lambda_n & & \\ & & \lambda_3-\lambda_n & \\ & & & \ddots \\ & & & & 0 \end{pmatrix}$$

$= O$ (對角線矩陣相乘 等於 對角線上各相對應項相乘).

By Problem 7,

$Q^{-1}p(A)Q = p(Q^{-1}AQ) = p(D) = O$.

$\Rightarrow p(A) = QOQ^{-1} = O$.