Sample Questions 15

Let $\mathbf{I}_n$ be the $n \times n$ identity matrix. Let $\mathbf{O}_n$ be the $n \times n$ zero matrix.

1. Let
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Compute $\mathbf{I}_4$, $\mathbf{A}$, $\mathbf{A}^2$, $\mathbf{A}^3$, and $\mathbf{A}^4$. Then find the minimal polynomial of $\mathbf{A}$. Also, find the minimal polynomial of $\mathbf{A} + \lambda \mathbf{I}_4$ for a given real number λ.

2. Let
$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$
Find the minimal polynomial of $\mathbf{A}$.

3. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by
$$f(\begin{bmatrix} r \\ \theta \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$
where $(x, y, z)^\top$ is the Cartesian coordinates of the point $(r, \theta, z)^\top$ in the cylinder coordinates. That is,
$$\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta, \\ z &= z. \end{aligned}$$
Find the Jacobian f' of f, and compute $\det(f')$.

4. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by
$$f(\begin{bmatrix} r \\ \theta \\ \phi \end{bmatrix}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$
where $(x, y, z)^\top$ is the Cartesian coordinates of the point $(r, \theta, \phi)^\top$ in the spherical coordinates. That is,
$$\begin{aligned} x &= r \sin \phi \cos \theta, \\ y &= r \sin \phi \sin \theta, \\ z &= r \cos \phi. \end{aligned}$$
Find the Jacobian f' of f, and compute $\det(f')$.

5. Define a map $f(\mathbf{v}) = \mathbf{A}\mathbf{v}$ with
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$
Find the Jacobian f' of f.

6. Suppose $f(x)$ is a polynomial over $\mathbb{C}$. Show that f as a multiple root if and only if $f'(x)$ and $f(x)$ has a common root. (A multiple root is a root c of $f(x)$ such th at $(x - c)^m$ is a factor of $f(x)$ with $m \geqslant 2$.)

7. Check if the polynomial
$$f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$$
has a multiple root in $\mathbb{C}$ or not.