Sample Solutions for Sample Questions 15.

1. $$I_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

$A^4 = O$.

Suppose $a_0 I_4 + a_1 A + a_2 A^2 + a_3 A^3 = O$

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & a_0 & a_1 & a_2 \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix} = O \Rightarrow a_0 = a_1 = a_2 = a_3 = 0.$$

$\Rightarrow \{I_4, A, A^2, A^3\}$ is linearly independent.

But $A^4 = 0$.

$\Rightarrow p(A) \neq 0$ if $\deg p(x) \leq 3$.

$p(A) = 0$ if $p(x) = x^4$. $\Rightarrow$ min poly is $p(x) = x^4$.

Similarly, the min poly of $A + \lambda I_4$

is $p(x) = (x - \lambda)^4$.

2.

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 4 & & & \\ & 4 & & \\ & & 9 & \\ & & & 9 \end{pmatrix}$$

Solve $\quad a_0 I + a_1 A + A^2 = O.$

$$\begin{pmatrix} a_0+2a_1+4 & & & \\ & a_0+2a_1+4 & & \\ & & a_0+3a_1+9 & \\ & & & a_0+3a_1+9 \end{pmatrix} = O.$$

$$\Rightarrow \begin{cases} a_0+2a_1 = -4 \\ a_0+3a_1 = -9 \end{cases} \Rightarrow \begin{cases} a_0 = \cancel{5}6 \\ a_1 = -5 \end{cases}$$

So $p(A)=0$ when $p(x) = x^2-5x+6 = (x-2)(x-3).$

(You may check $p(A) \neq 0$ if $\deg p(x) \leq 1$.)

So min poly is $p(x) = x^2-5x+6$.

3.
$$f' = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \det(f') = r\cos^2\theta + r\sin^2\theta = r.$$

4.
$$f' = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi \cancel{\cos} \sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

$$\cancel{\Rightarrow \det(f') = -r\sin\phi\,(\sin\phi\cos\theta\, r\sin\phi\cos\theta + r\sin\phi\sin\theta\sin\phi\sin\theta)}$$

$$\Rightarrow \det(f') = -r\sin\phi \cdot \det\begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta \end{pmatrix}$$
$$+ \cos\phi \cdot \det\begin{pmatrix} -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ r\sin\phi\cos\theta & r\cos\phi\sin\theta \end{pmatrix}$$
$$= -r^2(\sin\phi)^3 \cdot \det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + r^2(\cos\phi)^2\sin\phi \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$
$$= -r^2\sin\phi^3 - r^2(\cos\phi)^2\sin\phi$$
$$= -r^2\sin\phi\,(\sin^2\phi + \cos^2\phi) = -r^2\sin\phi.$$

5. $f(\vec{v}) = A\vec{v}$.

That is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\Rightarrow \begin{cases} y_1 = 1x_1 + 2x_2 + 3x_3 \\ y_2 = 4x_1 + 5x_2 + 6x_3 \end{cases}$$

$$f' = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

So $f' = A$.

In fact, if $f(\vec{v}) = A\vec{v}$,

then $f' = A$.

6.

Since $f(x)$ is over $\mathbb{C}$,

by the Fundamental Theorem of Algebra,

we may write

$$f(x) = (x-\lambda_1) \cdots (x-\lambda_n)$$

($\lambda_1, \ldots, \lambda_n$ might repeat).

① Suppose $\lambda_1, \ldots, \lambda_n$ ~~has~~ contains a multiple root.

Without loss of generality, assume $\lambda_1 = \lambda_2$.

Then $f'(x) = \dfrac{f(x)}{(x-\lambda_1)} + \dfrac{f(x)}{(x-\lambda_2)} + \cdots + \dfrac{f(x)}{x-\lambda_n}$

Note that $\cancel{\dfrac{f(x)}{x-\lambda_1}}$ $\dfrac{f(x)}{x-\lambda_1}$ contains the factor $x-\lambda_2$

$\dfrac{f(x)}{x-\lambda_2}$ — — — — $x-\lambda_1$

and $\dfrac{f(x)}{x-\lambda_i}$ has factors $x-\lambda_1$ and $x-\lambda_2$ for $i=3,\ldots,n$.

$\Rightarrow$ So $\cancel{f'(\lambda_1)}$ $\lambda_1=\lambda_2$ is a common root for $f(x)$ and $f'(x)$.

② Suppose $\lambda_1, \ldots, \lambda_n$ are all distinct.

Then $f'(x) = \dfrac{f(x)}{x-\lambda_1} + \dfrac{f(x)}{x-\lambda_2} + \cdots + \dfrac{f(x)}{x-\lambda_n}$.

Compute $f'(\lambda_1) = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3) \cdots (\lambda_1-\lambda_n) + 0 + 0 + \cdots + 0 \neq 0$.

Similarly, $f'(\lambda_2) \neq 0, \ldots, f'(\lambda_n) \neq 0$.

$\Rightarrow$ $f(x)$ and $f'(x)$ have no common root.

7.

Equivalently (By Problem 6),

we check if $f(x)$ and $f'(x)$ have a common root.

$$f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1 \qquad 4次 \qquad 4+3=7.$$

$$f'(x) = 4x^3 + 6x^2 + 6x + 2. \qquad 3次$$

①. ~~The~~ Sylvester matrix

$$S_{f,f'} = \begin{pmatrix} 1 & & & 2 & & & \\ 2 & 1 & & 6 & 2 & & \\ 3 & 2 & 1 & 6 & 6 & 2 & \\ 2 & 3 & 2 & 4 & 6 & 6 & 2 \\ 1 & 2 & 3 & 0 & 4 & 6 & 6 \\ 0 & 1 & 2 & 0 & 0 & 4 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \end{pmatrix} \qquad 7 \times 7$$

Use Sage or any method to compute & $\det(S_{f,f'}) = 0$

$\Rightarrow$ f has a multiple root.

②. 輾轉相除法.

$\frac{1}{4}x$	$x^4+2x^3+3x^2+2x+1$	$4x^3+6x^2+6x+2$.	$\frac{16}{3}x$
	$x^4+\frac{3}{2}x^3+\frac{3}{2}x^2+\frac{1}{2}x$	$4x^3+4x^2+4x$	
$\frac{1}{8}$	$\frac{1}{2}x^3+\frac{3}{2}x^2+\frac{3}{2}x+1$	$2x^2+2x+2$	$\frac{8}{3}$
$\cancel{8}$	$\frac{1}{2}x^3+\frac{3}{4}x^2+\frac{3}{4}x+\frac{1}{4}$	$2x^2+2x+2$	
	$\frac{3}{4}x^2+\frac{3}{4}x+\frac{3}{4}$	0.	

$\Rightarrow$ x^2+x+1 is a common factor of f and f'.