

Sample Solutions for Sample Questions 1.

1. There are many possible bases. The ones provided here are the "standard" ones.
(kind of)

$$(a) \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$(b) \{ 1, x, x^2, x^3 \}$$

$$(c) \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$(d) \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right. \\ \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$(e) \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

2.

$$(a). \quad \vec{v} = 3 \cdot 1 + 4 \cdot (x-1) + 5 \cdot (x-1)^2$$

[除非題目需要才要展開]

$$(b) \quad \vec{v} = 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 5 \\ 5 & 4 \end{pmatrix}$$

$$(c) \quad \vec{v} = 3 \cdot \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 19 \\ 55 \end{pmatrix}$$

3. (a) Find c_1, c_2, c_3 such that

$$x^2 = c_1 \cdot 1 + c_2 \cdot (x-1) + c_3 (x-1)^2$$

$$= c_1 \cdot 1$$

$$+ (-c_2) \cdot 1 + c_2 \cdot x$$

$$+ c_3 \cdot 1 + (-2c_3)x + c_3 x^2$$

$$\Rightarrow \begin{cases} c_1 - c_2 + c_3 = 0 \\ c_2 - 2c_3 = 0 \\ c_3 = 1 \end{cases} \Rightarrow \text{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

3 (b).

$$\begin{pmatrix} 5 & 7 \\ 7 & 3 \end{pmatrix} = 5 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 7 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{\mathcal{B}}(\vec{v}) = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}.$$

(c) Solve $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

$$\Rightarrow \text{Rep}_{\mathcal{B}}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 15 \end{pmatrix}$$

4. Find a representation of $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ by $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$

$$\Rightarrow \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So $f\left(\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right)$

$$= f\left(3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 3 \cdot f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + (-1) f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$$= 3 \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} + (-1) \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

5. Suppose $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$.

$$A e_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Similarly, $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

6. By 3(c), $\text{Rep}_B(\vec{e}_1) = \begin{pmatrix} 1 \\ -5 \\ 15 \end{pmatrix}$.

$$\text{Solve } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow c_1 = 0, c_2 = 1, c_3 = -5.$$

$$\Rightarrow \text{Rep}_B(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$$

$$\text{Solve } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow c_1 = c_2 = 0, c_3 = 1.$$

$$\Rightarrow \text{Rep}_B(\vec{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

7. Find $f(\vec{e}_1)$, $f(\vec{e}_2)$, and $f(\vec{e}_3)$.

$$\text{Then } A = \begin{pmatrix} f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \end{pmatrix}.$$

By Problem 6,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} - 5 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + 15 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{f} 1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} - 5 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 15 \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ = \begin{pmatrix} 33 \\ 44 \end{pmatrix}.$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} - 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{f} 0 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} - 5 \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ = \begin{pmatrix} -12 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{f} = 0 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\Rightarrow A = \begin{pmatrix} 33 & -12 & 3 \\ 44 & -16 & 4 \end{pmatrix}.$$

You may double-check that

$$A \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

