Sample Questions 2

Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 10 \\ 1 \end{bmatrix}$$

and

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a homomorphism such that

$$f(\mathbf{v}_1) = f(\mathbf{v}_2) = f(\mathbf{v}_3) = \mathbf{u}_1.$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$ and $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2\}$ a basis of $\mathbb{R}^2$. Also, let $\mathcal{S}_n$ be the standard basis of $\mathbb{R}^n$.

1. Find a matrx $\mathbf{A}$ such that $f(\mathbf{v}) = \mathbf{A}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

2. Find $\text{Rep}_{\mathcal{S}_3,\mathcal{S}_2}(f)$.

3. Find $\text{Rep}_{\mathcal{S}_3,\mathcal{D}}(f)$.

4. Find $\text{Rep}_{\mathcal{B},\mathcal{S}_2}(f)$.

5. Find $\text{Rep}_{\mathcal{B},\mathcal{D}}(f)$.

6. Let $\mathbf{B} = \text{Rep}_{\mathcal{B},\mathcal{D}}(f)$. You may check

$$\mathbf{B} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{D}} \text{ and } \mathbf{B} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{\mathcal{D}}.$$

   Explain the meaning of these two equality in terms of the homomorphism f.

7. Find the range and the rank of f. Find the null space and the nullity of f. (See Chapter Three.II.2 of the textbook for the definitions of the range and the null space.)