Sample Solutions for Sample Questions 2.

1. Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $A = \begin{pmatrix} | & | & | \\ f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \end{pmatrix}$

$$\vec{e}_1 = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3$$

$$\Rightarrow f(\vec{e}_1) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_1 + 0 \cdot \vec{u}_1 = \vec{u}_1$$

$$\vec{e}_2 = -10 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3$$

$$\Rightarrow f(\vec{e}_2) = -10 \cdot \vec{u}_1 + 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_1 = -9 \vec{u}_1$$

$$\vec{e}_3 = 95 \vec{v}_1 - 10 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3$$

$$\Rightarrow f(\vec{e}_3) = 95 \vec{u}_1 - 10 \vec{u}_1 + 1 \cdot \vec{u}_1 = 86 \vec{u}_1$$

So $A = \begin{pmatrix} | & | & | \\ \vec{u}_1 & -9\vec{u}_1 & 86\vec{u}_1 \\ | & | & | \end{pmatrix}$

$$= \begin{pmatrix} 3 & -27 & 258 \\ 4 & -36 & 344 \end{pmatrix}.$$

2. Let $S_3 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Then $$\mathrm{Rep}_{S_3,S_2}(f) = \begin{pmatrix} | & | & | \\ \mathrm{Rep}_{S_2}(f(\vec{e}_1)) & \mathrm{Rep}_{S_2}(f(\vec{e}_2)) & \mathrm{Rep}_{S_2}(f(\vec{e}_3)) \\ | & | & | \end{pmatrix}$$

Note that $\mathrm{Rep}_{S_2}(\vec{w}) = \vec{w}$ for all $\vec{w} \in \mathbb{R}^2$.

So $$\mathrm{Rep}_{S_3,S_2}(f) = \begin{pmatrix} | & | & | \\ f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \end{pmatrix}$$

$= A$ in Problem 1.

(This is not a coincidence.).

3. Let $S_3 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

Then $\mathrm{Rep}_{S_3, D}(f) = \begin{pmatrix} | & | & | \\ \mathrm{Rep}_D(f(\vec{e}_1)) & \mathrm{Rep}_D(f(\vec{e}_2)) & \mathrm{Rep}_D(f(\vec{e}_3)) \\ | & | & | \end{pmatrix}$.

We already computed $f(\vec{e}_1)$, $f(\vec{e}_2)$, $f(\vec{e}_3)$ in Problem 1.

$f(\vec{e}_1) = \vec{u}_1 + 0\vec{u}_2 \Rightarrow \mathrm{Rep}_D(f(\vec{e}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f(\vec{e}_2) = -9\,\vec{u}_1 + 0\vec{u}_2 \Rightarrow \mathrm{Rep}_D(f(\vec{e}_3)) = \begin{pmatrix} -9 \\ 0 \end{pmatrix}$

$f(\vec{e}_3) = 86\,\vec{u}_1 + 0\vec{u}_2 \Rightarrow \mathrm{Rep}_D(f(\vec{e}_3)) = \begin{pmatrix} 86 \\ 0 \end{pmatrix}$.

$\Rightarrow \mathrm{Rep}_{S_3, D}(f) = \begin{pmatrix} 1 & -9 & 86 \\ 0 & 0 & 0 \end{pmatrix}$.

4. $B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$.

Then $\operatorname{Rep}_{B,S_2}(f) = \begin{pmatrix} | & | & | \\ \operatorname{Rep}_{S_2}(f(\vec{v}_1)) & \operatorname{Rep}_{S_2}(f(\vec{v}_2)) & \operatorname{Rep}_{S_2}(f(\vec{v}_3)) \\ | & | & | \end{pmatrix}$

Again, $\operatorname{Rep}_{S_2}(\vec{w}) = \vec{w}$ for all $\vec{w} \in \mathbb{R}^2$.

$$\operatorname{Rep}_{B,S_2}(f) = \begin{pmatrix} | & | & | \\ f(\vec{v}_1) & f(\vec{v}_2) & f(\vec{v}_3) \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_1 & \vec{u}_1 \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \end{pmatrix}.$$

5. $B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$.

Then $\mathrm{Rep}_{B,D}(f) = \begin{pmatrix} | & | & | \\ \mathrm{Rep}_D(f(\vec{v}_1)) & \mathrm{Rep}_D(f(\vec{v}_2)) & \mathrm{Rep}_D(f(\vec{v}_3)) \\ | & | & | \end{pmatrix}$.

It's super easy.

$f(\vec{v}_1) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \Rightarrow \mathrm{Rep}_D(f(\vec{v}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f(\vec{v}_2) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \Rightarrow \mathrm{Rep}_D((f(\vec{v}_2)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f(\vec{v}_3) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \Rightarrow \mathrm{Rep}_D((f(\vec{v}_3)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

So. $\mathrm{Rep}_{B,D}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

6.

[By Problem 5, $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

You may check $B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $B\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.].

$B\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{D}}$ means.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} \xmapsto{B \cdot \square} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{D}}$$

and

$$1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 \xmapsto{f} 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2.$$

That is, $f(\vec{v}_1) = \vec{u}_1$.

$B\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\mathcal{D}}$ means

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \xmapsto{B \cdot \square} \begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\mathcal{D}}$$

$$1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 \xmapsto{f} 2\vec{u}_1 + 0 \cdot \vec{u}_2$$

That is, $f(\vec{v}_1 + \vec{v}_2) = 2\vec{u}_1$.

~~1~~ 7. Use $A = \begin{pmatrix} 3 & -27 & 258 \\ 4 & -36 & 344 \end{pmatrix}$ in Problem 1.

$$\text{range}(f) \overset{\text{def}}{=\!=} \{ f(\vec{v}) \mid \vec{v} \in \mathbb{R}^3 \}.$$

$$= \{ A\vec{v} \mid \vec{v} \in \mathbb{R}^3 \}$$

$$= \text{Columspace}(A)$$

$$= \text{span}\left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}. \longrightarrow \text{rank} = 1.$$

$$\text{nullspace}(f) \overset{\text{def}}{=\!=} \{ \vec{v} \in \mathbb{R}^3 \mid f(\vec{v}) = \vec{0} \in \mathbb{R}^2 \}.$$

$$= \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{0} \in \mathbb{R}^2 \}$$

$$= \text{nullspace}(A)$$

$$= \text{span}\left\{ \begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -86 \\ 0 \\ 1 \end{pmatrix} \right\} \longrightarrow \text{nullity} = 2.$$

Check Sample Question 8 in Math 103 (and its solution) to see how to compute the basis of ~~range~~, column space, row space, and nul space.