Sample Questions 3

For all problems, let $\mathcal{M}_{m\times n}$ be the space of all 2×3 matrices, and let $\mathcal{P}_n$ be the space of all polynomials of degree at most n.

Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i,j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

1. Let $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(\mathbf{v})$.

2. Find a matrix $\mathbf{v}$ such that $f(\mathbf{v}) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ or show such a matrix does not exist.

3. Find the nullspace and the nullity of f.

4. Find the range and the rank of f.

Let $\mathcal{B} = \{1, x\}$ be a basis of $\mathcal{P}_2$ and $\mathcal{D} = \{1, x, x^2\}$ be a basis of $\mathcal{P}_3$. Define $f : \mathcal{P}_2 \to \mathcal{P}_3$ by $f(p(x)) = (x+1)\cdot p(x)$. Define $g : \mathcal{P}_2 \to \mathcal{P}_3$ by $g(p(x)) = (x-1)\cdot p(x)$.

5. Check that $f+g$ is also a homomorphism from $\mathcal{P}_2$ to $\mathcal{P}_3$ defined by $(f+g)(p(x)) = (2x)\cdot p(x)$. Let $\mathbf{A} = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f)$, $\mathbf{B} = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(g)$, and $\mathbf{C} = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f+g)$. Find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and then check if $\mathbf{A}+\mathbf{B} = \mathbf{C}$. Is $f+g$ one-to-one? Is it onto?

Let $\mathcal{B}_n = \{1, \ldots, x^n\}$ be a basis of $\mathcal{P}_n$. Define $f : \mathcal{P}_3 \to \mathcal{P}_2$ by $f(p(x)) = p'(x)$. Define $g : \mathcal{P}_2 \to \mathcal{P}_1$ by $g(p(x)) = p'(x)$. That is, both f and g are the differential operator, with different domains and different codomains.

6. Check that $g\circ f$ is a homomorphism from $\mathcal{P}_3$ to $\mathcal{P}_1$ defined by $g\circ f(p(x)) = p''(x)$. Let $\mathbf{A} = \mathrm{Rep}_{\mathcal{B}_3,\mathcal{B}_2}(f)$, $\mathbf{B} = \mathrm{Rep}_{\mathcal{B}_2,\mathcal{B}_1}(g)$, and $\mathbf{C} = \mathrm{Rep}_{\mathcal{B}_3,\mathcal{B}_1}(g\circ f)$. Find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and then check if $\mathbf{BA} = \mathbf{C}$. Is $g\circ f$ one-to-one? Is it onto?

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7. Define $\mathrm{id} : \mathcal{P}_2 \to \mathcal{P}_2$ by $\mathrm{id}(p(x)) = p(x)$. Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{D} = \{1, x+1, (x+1)^2\}$ be two bases of $\mathcal{P}$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(\mathrm{id})$. Is id nonsingular?