

Sample Solutions for Sample Questions 3.

1. $\text{Rep}_B(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

Compute $A \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 4 \\ 5 \end{pmatrix}$

So $f(\vec{v}) = 1E_{12} + 2E_{13} + 4E_{22} + 5E_{23}$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 4 & 5 \end{pmatrix}$$

2. No such matrix exist.

~~$\text{Rep}_B(\vec{v})$~~ $\text{Rep}_B \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

To find $f(\vec{v}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

need to find $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ such that $\text{Rep}_B(\vec{v}) = \vec{x}$.

But $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ has no solution.

So such \vec{v} does not exist.

3.

Solve $A\vec{x} = \vec{0}$.

$$A \rightsquigarrow \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix} \end{matrix}$$

↑ ↑
free variables.

Find nullspace (A):

- set $x_3=1, x_6=0 \Rightarrow$ solve $A\vec{\beta}_1 = \vec{0}$

and get $\vec{\beta}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

- set $x_3=0, x_6=1 \Rightarrow$ solve $A\vec{\beta}_2 = \vec{0}$

and get $\vec{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\rightarrow \text{nullspace}(A) = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Therefore,

$$\text{nullspace}(f) = \text{span} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \Rightarrow \text{nullity} = 2$$

$$= \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

4. $\text{range}(f)$ corresponds to the column space of A .

$$A \rightsquigarrow \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix} \end{matrix}$$

$\uparrow \uparrow$ $\uparrow \uparrow$
 $x_1 \ x_2$ $x_4 \ x_5$ leading variables.

So

$$\text{Colspace}(A) = \{ \text{1st, 2nd, 4th, 5th columns of } A \}$$

$$= \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Therefore,

$$\text{range}(f) = \text{Span} \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \Rightarrow \text{rank} = 4.$$

\uparrow Rep B

$$= \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

That's also why $f(\vec{v}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ has

no solution, since $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is not in $\text{range}(f)$.

5. ① Check .

$$\begin{aligned}(f+g)(p(x)) &= f(p(x)) + g(p(x)) \\ &= (x+1) \cdot p(x) + (x-1) p(x) \\ &= 2x \cdot p(x).\end{aligned}$$

$$\begin{aligned}\textcircled{2}. \text{ Find } A &= \begin{pmatrix} \text{Rep}_D(f(1)) & \text{Rep}_D(f(x)) \\ | & | \\ \text{Rep}_D(x+1) & \text{Rep}_D(x^2+x) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ | & | \\ 1 & 1 \\ | & | \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\textcircled{3}. \text{ Find } B &= \begin{pmatrix} \text{Rep}_D(g(1)) & \text{Rep}_D(g(x)) \\ | & | \\ \text{Rep}_D(x-1) & \text{Rep}_D(x^2-x) \\ | & | \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ | & | \\ 1 & -1 \\ | & | \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\textcircled{4}. \text{ Find } C &= \begin{pmatrix} \text{Rep}_D((f+g)(1)) & \text{Rep}_D((f+g)(x)) \\ | & | \\ \text{Rep}_D(2x) & \text{Rep}_D(2x^2) \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ | & | \\ 2 & 0 \\ | & | \\ 0 & 2 \end{pmatrix}.\end{aligned}$$

⑤ Check $A+B=C$. True.

5. [Conti.] ^⑥ one-to-one? onto?

$$\text{Rep}_{B_2, D}(f+g) = C = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ has rank } 2.$$

rank = ~~2~~ # of columns \Rightarrow one-to-one

rank \neq # of ~~columns~~ ^{rows} \Rightarrow not onto.

6. Φ check:

$$g \circ f(p(x)) = g(f(p(x))) = g(p'(x)) = p''(x)$$

$$\begin{aligned} \textcircled{2} \text{ Find } A &= \left(\begin{array}{c|c|c|c} \text{Rep}_{B_2}(f(1)) & \text{Rep}_{B_2}(f(x)) & \text{Rep}_{B_2}(f(x^2)) & \text{Rep}_{B_2}(f(x^3)) \\ \hline | & | & | & | \end{array} \right) \\ &= \left(\begin{array}{c|c|c|c} \text{Rep}_{B_2}(0) & \text{Rep}_{B_2}(1) & \text{Rep}_{B_2}(2x) & \text{Rep}_{B_2}(3x^2) \\ \hline | & | & | & | \end{array} \right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \textcircled{3} \text{ Find } B &= \left(\begin{array}{c|c|c} \text{Rep}_{B_1}(g(1)) & \text{Rep}_{B_1}(g(x)) & \text{Rep}_{B_1}(g(x^2)) \\ \hline | & | & | \end{array} \right) \\ &= \left(\begin{array}{c|c|c} \text{Rep}_{B_1}(0) & \text{Rep}_{B_1}(1) & \text{Rep}_{B_1}(2x) \\ \hline | & | & | \end{array} \right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \textcircled{4} \text{ Find } C &= \left(\begin{array}{c|c|c|c} \text{Rep}_{B_1}(g(f(1))) & \text{Rep}_{B_1}(g(f(x))) & \text{Rep}_{B_1}(g(f(x^2))) & \text{Rep}_{B_1}(g(f(x^3))) \\ \hline | & | & | & | \end{array} \right) \\ &= \left(\begin{array}{c|c|c|c} \text{Rep}_{B_1}(0) & \text{Rep}_{B_1}(0) & \text{Rep}_{B_1}(2) & \text{Rep}_{B_1}(6x) \\ \hline | & | & | & | \end{array} \right) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}. \end{aligned}$$

$\textcircled{5}$ ~~A~~ check $BA = C$. True.

6. [Conti.]

⑥ one-to-one? onto?

$$\text{Rep}_{B_2, B_1}(g \circ f) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \text{ has rank } 2.$$

rank \neq # of columns \Rightarrow not one-to-one

rank = # of rows \Rightarrow onto.

$$\begin{aligned} 7. \text{ Let } A &= \text{Rep}_{B, D}(\text{id}) = \begin{pmatrix} | & | & | \\ \text{Rep}_D(\text{id}(1)) & \text{Rep}_D(\text{id}(x)) & \text{Rep}_D(\text{id}(x^2)) \\ | & | & | \end{pmatrix} \\ &= \begin{pmatrix} | & | & | \\ \text{Rep}_D(1) & \text{Rep}_D(x) & \text{Rep}_D(x^2) \\ | & | & | \end{pmatrix}. \end{aligned}$$

$$\text{Compute } 1 = 1 \cdot 1 + 0 \cdot (x+1) + 0 \cdot (x+1)^2$$

$$x = -1 \cdot 1 + 1 \cdot (x+1) + 0 \cdot (x+1)^2$$

$$x^2 = 1 \cdot 1 - 2(x+1) + 1 \cdot (x+1)^2$$

$$\Rightarrow A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

A is nonsingular \Rightarrow ~~f~~ id is nonsingular.

$$\left(\begin{array}{l} \text{rank} = \# \text{ of columns} \\ = \# \text{ of rows} \\ = 3 \end{array} \right)$$

(Indeed, id is of course one-to-one and onto, so it is nonsingular.)