Sample Questions 4

Let $\mathcal{P}_n$ be the polynomials of degree at most n. Let $\mathcal{S}_n$ be the standard basis of $\mathbb{R}^n$. Let $\mathbf{I}_n$ be the identity matrix of order n. Let $\mathbf{J}_n$ be the all-ones matrix of order n.

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1. Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{D} = \{1, x, x(x-1)\}$ be two bases of $\mathcal{P}_2$. Find matrices M and N such that

$$\mathbf{M}\operatorname{Rep}_{\mathcal{B}}(\mathbf{p}) = \operatorname{Rep}_{\mathcal{D}}(\mathbf{p}) \text{ and}$$
$$\mathbf{N}\operatorname{Rep}_{\mathcal{D}}(\mathbf{p}) = \operatorname{Rep}_{\mathcal{B}}(\mathbf{p})$$

for all $\mathbf{p} \in \mathcal{P}_2$. Also, check that if $\mathbf{MN} = \mathbf{NM} = \mathbf{I}_3$ or not.

2. Let $\mathcal{B} = \mathcal{S}_3$ and

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be two bases of $\mathbb{R}^3$. Find matrices $\mathbf{M}$ and $\mathbf{N}$ such that

$$\mathbf{M}\operatorname{Rep}_{\mathcal{B}}(\mathbf{v}) = \operatorname{Rep}_{\mathcal{D}}(\mathbf{v}) \text{ and}$$
$$\mathbf{N}\operatorname{Rep}_{\mathcal{D}}(\mathbf{v}) = \operatorname{Rep}_{\mathcal{B}}(\mathbf{v})$$

for all $\mathbf{v} \in \mathcal{P}_2$. Also, check that if $\mathbf{MN} = \mathbf{NM} = \mathbf{I}_3$ or not.

Let $\mathcal{B} = \mathcal{S}_2$ and

$$\mathcal{D} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

be two bases of $\mathbb{R}^2$. Then there is a relation between

$$\begin{bmatrix} x \\ y \end{bmatrix} = \operatorname{Rep}_{\mathcal{B}}(\mathbf{v}) \text{ and } \begin{bmatrix} x' \\ y' \end{bmatrix} = \operatorname{Rep}_{\mathcal{D}}(\mathbf{v}).$$

3. On the $\mathbb{R}^2$ plane, the equation $(x + y)(x - y) = 0$ is a diagonal cross. Rewrite the equation using x' and y'. Use desmos to see the figures of the two equations.

4. On the $\mathbb{R}^2$ plane, the equation $7x^2 - 2xy + 7y^2 = 1$ is an ellipse. Rewrite the equation using x' and y'. Use desmos to see the figures of the two equations.

5. Define a homomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(\mathbf{v}) = \mathbf{J}_2\mathbf{v}$. It is easy to see that $\mathbf{J}_2 = \operatorname{Rep}_{\mathcal{S}_2,\mathcal{S}_2}(f)$. Instead of using the standard basis $\mathcal{S}_2$, find $\Lambda = \operatorname{Rep}_{\mathcal{D},\mathcal{D}}(f)$. Try to describe the geometry of f.

6. Let Λ be as in the previous problem. Find $\mathbf{Q} = \operatorname{Rep}_{\mathcal{D},\mathcal{S}_2}(\mathrm{id})$. Also, check that if $\mathbf{Q}^{-1}\mathbf{J}_2\mathbf{Q} = \Lambda$. Is this a coincidence?

7. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 4 & 6 & 10 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be two matrices. Find $\mathbf{P}$ and $\mathbf{Q}$ so that $\mathbf{PAQ} = \mathbf{B}$.