Sample Solutions for Sample Questions 4.

1. $M = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(id) = \begin{pmatrix} | & | & | \\ \mathrm{Rep}_{D}(1) & \mathrm{Rep}_{D}(x) & \mathrm{Rep}_{D}(x^2) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

The entries come from:

$1 = 1\cdot 1 + 0\cdot x + 0\cdot x(x-1)$

$x = 0\cdot 1 + 1\cdot x + 0\cdot x(x-1)$

$x^2 = 0\cdot 1 + 1\cdot x + 1\cdot x(x-1)$

$N = \mathrm{Rep}_{\mathcal{D},\mathcal{B}}(id) = \begin{pmatrix} | & | & | \\ \mathrm{Rep}_{\mathcal{B}}(1) & \mathrm{Rep}_{\mathcal{B}}(x) & \mathrm{Rep}_{\mathcal{B}}(x(x-1)) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

The entries come from:

$1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$x = 0\cdot 1 + 1\cdot x + 0\cdot x^2$

$x(x-1) = 0\cdot 1 - 1\cdot x + 1\cdot x^2$

Check: $MN = NM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. True.

2.

$$M = \mathrm{Rep}_{B,D}(id) = \begin{pmatrix} \vert & \vert & \vert \\ \mathrm{Rep}_D\begin{pmatrix}1\\0\\0\end{pmatrix} & \mathrm{Rep}_D\begin{pmatrix}0\\1\\0\end{pmatrix} & \mathrm{Rep}_D\begin{pmatrix}0\\0\\1\end{pmatrix} \\ \vert & \vert & \vert \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The entries come from:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = -1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + (-1)\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$$N = \mathrm{Rep}_{\cancel{B},D,B}(id) = \begin{pmatrix} \vert & \vert & \vert \\ \mathrm{Rep}_B\begin{pmatrix}1\\0\\0\end{pmatrix} & \mathrm{Rep}_B\begin{pmatrix}1\\1\\0\end{pmatrix} & \mathrm{Rep}_B\begin{pmatrix}1\\1\\1\end{pmatrix} \\ \vert & \vert & \vert \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The entries come from:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Check: $MN = NM = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ True.

※. Before Problem 3, let's find the relations between $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$.

① Let $M = \mathrm{Rep}_{B,D}(id) = \left(\mathrm{Rep}_D \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \ \mathrm{Rep}_D \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

The entries come from:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \left(-\frac{1}{\sqrt{2}}\right) \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Then $M \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{cases} x' = \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y & \cdots (1) \\ y' = -\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y. \end{cases}$

② Let $N = \mathrm{Rep}_{D,B}(id) = \left(\cancel{\mathrm{Rep}_B(id)} \right) = \left(\mathrm{Rep}_B \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \ \ \mathrm{Rep}_B \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$\mathrm{Rep}_B(\vec{v}) = \vec{v}$ for all $\vec{v} \in R^2$

since B is the standard basis

Then $N \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x = \frac{1}{\sqrt{2}} x' - \frac{1}{\sqrt{2}} y' & \cdots (2) \\ y = \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y'. \end{cases}$

3. Use Equation (2) on page 3.

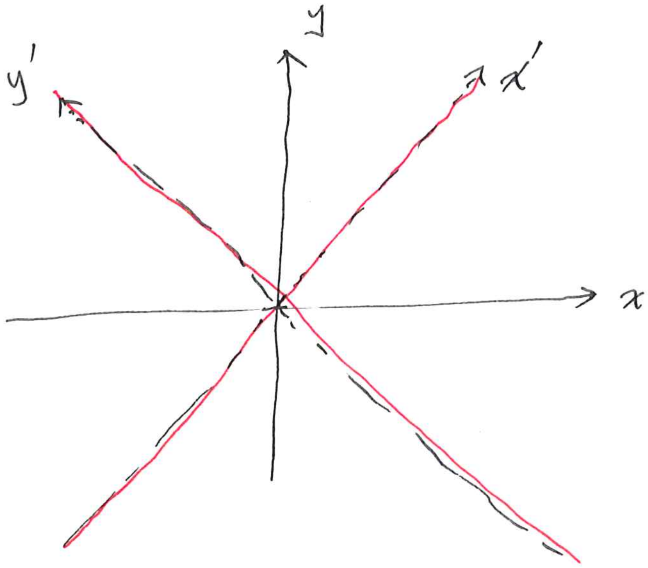
$$x+y=\left(\frac{1}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'\right)+\left(\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\right)=\sqrt{2}\,x'$$

$$x-y=\left(\frac{1}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'\right)-\left(\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\right)=-\sqrt{2}\,y'.$$

So the new equation is $(\sqrt{2}\,x')(-\sqrt{2}\,y')=0$

$$\Updownarrow$$

$$-2x'y'=0.$$



The red figure is

$(x+y)(x-y)=0$

on xy-plane,

and is

$-2x'y'=0$

on $x'y'$-plane.

4. Again, use ~~eq~~ Equation (2) on page 3.

$$x^2+y^2=\left(\frac{1}{2}x'^2-x'y'+\frac{1}{2}y'^2\right)+\left(\frac{1}{2}x'^2+x'y'+\frac{1}{2}y'^2\right)$$

$$=x'^2+y'^2$$ (not surprising, since two coordinating systems has the same radius.)

$$xy=\frac{1}{2}x'^2-\frac{1}{2}y'^2$$

So $7x^2-2xy+7y^2=\cancel{0}1$ ~~be cor~~ becomes $7(x'^2+y'^2)-2\left(\frac{1}{2}x'^2-\frac{1}{2}y'^2\right)=\cancel{0}1.$

$$\Leftrightarrow\ 6x'^2+8y'^2=\cancel{0}1.$$

Use desmos.com to see the ellipse.

5.

By definition,

$$\mathrm{Rep}_{D,D}(f) = \left(\mathrm{Rep}_D\left(f\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\right) \quad \mathrm{Rep}_D\left(f\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\right) \right).$$

Computation:

$$f\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = J_2\cdot\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\\sqrt{2}\end{pmatrix} = 2\cdot\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} + 0\cdot\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}.$$

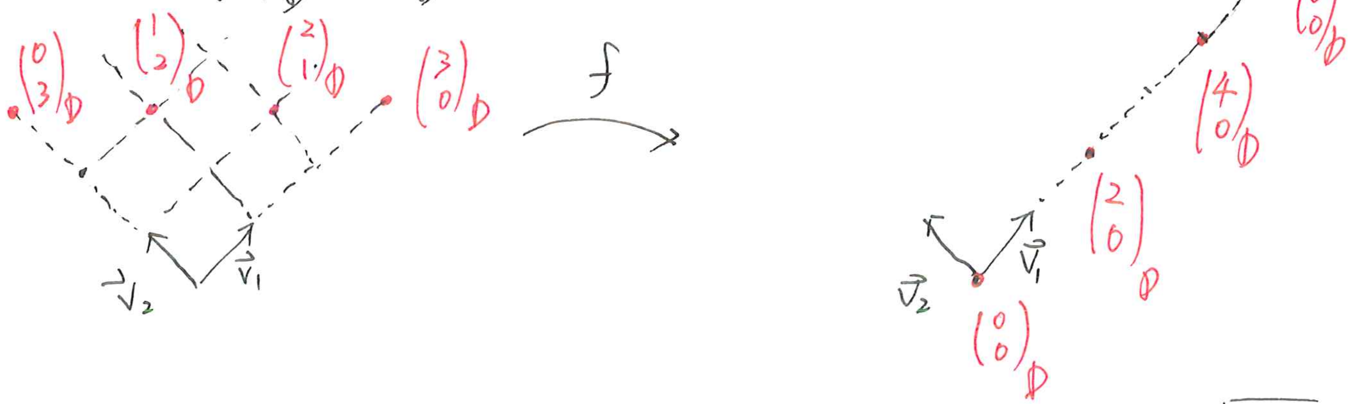
$$f\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = J_2\cdot\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} + 0\cdot\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}.$$

$$\Rightarrow \mathrm{Rep}_{D,D}(f) = \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix} = \Lambda.$$

Let $\vec{v}_1 = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

$\mathrm{Rep}_{D,D}(f)\cdot\begin{pmatrix}1\\0\end{pmatrix}_D = \begin{pmatrix}2\\0\end{pmatrix}_D$ means $\vec{v}_1 \xmapsto{f} 2\vec{v}_1$

$\mathrm{Rep}_{D,D}(f)\cdot\begin{pmatrix}0\\1\end{pmatrix}_D = \begin{pmatrix}0\\0\end{pmatrix}_D$ means $\vec{v}_2 \xmapsto{f} \vec{0}$



6.

In fact, we already computed

$$Q = Rep_{D,S_2}(id) = N \text{ on page 3.}$$

$$Q^{-1} = Rep_{S_2,D}(id) = M \text{ on page 3.}$$

You may check

$$Q^{-1} J_2 Q = \Lambda \text{ on page 5.}$$

This is not a coincidence.

$$\Lambda = Rep_{D,D}(f)$$

$$= \underbrace{Rep_{S_2,D}(id)}_{Q^{-1}} \cdot \underbrace{Rep_{S_2,S_2}(f)}_{J_2} \cdot \underbrace{Rep_{D,S_2}(id)}_{Q}$$

1.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 4 & 6 & 10 \end{pmatrix} \xrightarrow[-4r_1+r_3]{-3r_1+r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[-3c_1+c_3]{-2c_1+c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-c_2+c_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = B.$$

$$\begin{array}{ccc} -\frac{1}{2}r_2 & -r_2+r_3 & \begin{array}{c}-3r_1+r_2 \\ -4r_1+r_3\end{array} \end{array}$$

Let $P = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -3 & 1 & \\ -4 & & 1 \end{pmatrix}$ 注意順序.

$Q = \cancel{\begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix}} \begin{pmatrix} 1 & -2 & -3 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{pmatrix}$

$$\begin{array}{cc} \begin{array}{c}-2c_1+c_2 \\ -3c_1+c_3\end{array} & -c_2+c_3 \end{array}$$

You may compute $P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & -2 & \cancel{-3}\,-1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

and check $PAQ = B$.

愈接近 A 的矩陣愈先發生.