Sample Questions 5

Let $\mathcal{M}_{m\times n}$ be the space of all $m \times n$ matrices. Let $\mathcal{P}_n$ be the polynomials of degree at most n. Let $\mathcal{S}_n$ be the standard basis of $\mathbb{R}^n$. Let $\mathbf{I}_n$ be the identity matrix of order n. Let $\mathbf{J}_n$ be the all-ones matrix of order n.

Let $E_{i,j}$ be the 2×2 matrix whose i,j-entry is 1 while other entries are zeros. Then $\mathcal{B} = \{\mathbf{E}_{1,1}, \mathbf{E}_{1,2}, \mathbf{E}_{2,1}, \mathbf{E}_{2,2}\}$ is a basis of $\mathcal{M}_{2\times 2}$.

1. Let $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be a homomorphism defined by $f(\mathbf{A}) = \mathbf{J}_2\mathbf{A}$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

2. Let $f : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be a homomorphism defined by $f(\mathbf{A}) = \mathbf{A}^\top$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

Let $\mathcal{B}_n = \{1, \dots, x^n\}$ be a basis of $\mathcal{P}_n$.

3. Let $f : \mathcal{P}_2 \to \mathcal{P}_1$ be a homomorphism defined by $f(\mathbf{p}(x)) = \mathbf{p}'(x)$. Let $g : \mathcal{P}_1 \to \mathcal{P}_2$ be a homomorphism defined by $g(\mathbf{p}(x)) = \mathbf{q}(x)$ with $\mathbf{q}'(x) = \mathbf{p}(x)$ and $\mathbf{q}(0) = 0$. Find $\mathrm{Rep}_{\mathcal{B}_2,\mathcal{B}_1}(f)$, $\mathrm{Rep}_{\mathcal{B}_1,\mathcal{B}_2}(g)$, and $\mathrm{Rep}_{\mathcal{B}_2,\mathcal{B}_2}(g \circ f)$.

4. Let $f : \mathcal{P}_2 \to \mathbb{R}^3$ be a homomorphism defined by
$$f(\mathbf{p}(x)) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \\ \mathbf{p}(3) \end{bmatrix}.$$
Find $\mathbf{M} = \mathrm{Rep}_{\mathcal{B}_2,\mathcal{S}_3}(f)$. (A matrix of this form is called a Vandermonde matrix.)

5. Let
$$\mathbf{p}_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)},$$
$$\mathbf{p}_2(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)},$$
$$\mathbf{p}_3(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)}.$$
(These are examples of Lagrange polynomials.) Let $g : \mathbb{R}^3 \to \mathcal{P}_2$ be a homomorphism defined by
$$g(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = a\mathbf{p}_1(x) + b\mathbf{p}_2(x) + c\mathbf{p}_3(x).$$
Find $\mathbf{N} = \mathrm{Rep}_{\mathcal{S}_3,\mathcal{B}_2}(g)$. Then check $\mathbf{MN} = \mathbf{NM} = \mathbf{I}_3$.

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6. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of a vector space V. Let $\lambda_1, \dots, \lambda_n$ be some real numbers. Suppose $f : V \to V$ is a homomorphism defined by $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.

7. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of a vector space V. Let λ be a real number. Suppose $f : V \to V$ is a homomorphism defined by $f(\mathbf{v}_1) = \lambda\mathbf{v}_1$ and $f(\mathbf{v}_i) = \lambda\mathbf{v}_i + \mathbf{v}_{i-1}$ for all $i \geqslant 2$. Find $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$.