Sample Solutions for Sample Questions 5

1.

$$f(E_{1,1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\mathrm{Rep}_B} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f(E_{1,2}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{Rep}_B} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$f(E_{2,1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\mathrm{Rep}_B} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f(E_{2,2}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{Rep}_B} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

So $\mathrm{Rep}_{B,B}(f) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$

2. $f(E_{1,1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T = E_{1,1} \xrightarrow{Rep_B} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$f(E_{1,2}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T = E_{2,1} \xrightarrow{Rep_B} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$f(E_{2,1}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T = E_{1,2} \xrightarrow{Rep_B} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$f(E_{2,2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T = E_{2,2} \xrightarrow{Rep_B} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

So $Rep_{B,B}(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3.

① Compute $Rep_{B_2, B_1}(f)$

$$f(1) = 0 \xrightarrow{Rep_{B_1}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f(x) = 1 \xrightarrow{Rep_{B_1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(x^2) = 2x \xrightarrow{Rep_{B_1}} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad Rep_{B_2, B_1}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

② Compute $Rep_{B_1, B_2}(g)$

$$g(1) = x \xrightarrow{Rep_{B_2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$g(x) = \frac{1}{2}x^2 \xrightarrow{Rep_{B_2}} \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow \quad Rep_{B_1, B_2}(g) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

③

$$Rep_{B_2, B_2}(g \circ f) = Rep_{B_1, B_2}(g) \cdot Rep_{B_2, B_1}(f)$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.

$$f(1) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{Rep}_{S_3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$f(x) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{\text{Rep}_{S_3}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$f(x^2) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix} \xrightarrow{\text{Rep}_{S_3}} \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}.$$

$$\Rightarrow \quad \text{Rep}_{B_2, S_3}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

M

5. Expand $p_1(x)$, $p_2(x)$, $p_3(x)$ first.

$$p_1(x) = \frac{1}{2}(x^2 - 5x + 6)$$

$$p_2(x) = -1(x^2 - 4x + 3)$$

$$p_3(x) = \frac{1}{2}(x^2 - 3x + 2)$$

Compute $\mathrm{Rep}_{S_3, B_2}(g)$

$$g\begin{pmatrix}1\\0\\0\end{pmatrix} = p_1(x) \xrightarrow{\mathrm{Rep}_{B_2}} \begin{array}{c} \\ \\ \\ \end{array}\begin{pmatrix}3\\-5/2\\1/2\end{pmatrix}\begin{array}{c}1\\x\\x^2\end{array}$$

Be aware of the order.

$$g\begin{pmatrix}0\\1\\0\end{pmatrix} = p_2(x) \xrightarrow{\mathrm{Rep}_{B_2}} \begin{pmatrix}-3\\4\\-1\end{pmatrix}$$

$$g\begin{pmatrix}0\\0\\1\end{pmatrix} = p_3(x) \xrightarrow{\mathrm{Rep}_{B_2}} \begin{pmatrix}1\\-3/2\\1/2\end{pmatrix}$$

N.

$$\Rightarrow \mathrm{Rep}_{S_3, B_2}(g) = \begin{pmatrix}3 & -3 & 1\\ -5/2 & 4 & -3/2\\ 1/2 & -1 & 1/2\end{pmatrix}$$

You may check $MN = NM = I_3$.

6.

$$f(\vec{v}_i) = \lambda_i \vec{v}_i \xrightarrow{Rep_B} \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{i-th entry.} \\ \\ \\ \\ \end{array} \end{array}$$

$$\Rightarrow Rep_{B,B}(f) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

diagonal matrix

7. $$f(\vec{v}_1) = \lambda \vec{v}_1 \xrightarrow{Rep_B} \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

For $i \geq 2$,

$$f(\vec{v}_i) = \lambda \vec{v}_i + \vec{v}_{i-1} \xrightarrow{Rep_B} \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{(i-1)-th entry} \\ \leftarrow \text{i-th entry.} \\ \\ \\ \\ \end{array} \end{array}$$

$$\Rightarrow Rep_{B,B}(f) = \begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

Called a Jordan block.