Sample Questions 6

1. Let
$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$
Apply the Gram–Schmidt algorithm to the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and obtain an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for the space $\mathbb{R}^3 = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

2. Following the previous problem, normalize the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to obtain an orthonormal basis $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$. Going through all the calculation, you should be able to find
$$\begin{aligned}\mathbf{v}_1 &= r_{1,1}\hat{\mathbf{u}}_1 + r_{2,1}\hat{\mathbf{u}}_2 + r_{3,1}\hat{\mathbf{u}}_3,\\ \mathbf{v}_2 &= r_{1,2}\hat{\mathbf{u}}_1 + r_{2,2}\hat{\mathbf{u}}_2 + r_{3,2}\hat{\mathbf{u}}_3,\\ \mathbf{v}_3 &= r_{1,3}\hat{\mathbf{u}}_1 + r_{2,3}\hat{\mathbf{u}}_2 + r_{3,3}\hat{\mathbf{u}}_3.\end{aligned}$$
Let
$$\mathbf{A} = \begin{bmatrix}| & | & |\\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\\ | & | & |\end{bmatrix}, \mathbf{Q} = \begin{bmatrix}| & | & |\\ \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3\\ | & | & |\end{bmatrix}$$
and $\mathbf{R} = \begin{bmatrix}r_{i,j}\end{bmatrix}$. Verify that $\mathbf{A} = \mathbf{QR}$ such that $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

3. Let $V = \{\begin{bmatrix}x\\y\\z\end{bmatrix} : x + y + z = 0\}$ be a vector space. Find an orthogonal basis for V.

4. Recall that an $n \times n$ matrix is an orthogonal matrix if $\mathbf{A}^\top\mathbf{A} = \mathbf{A}\mathbf{A}^\top = \mathbf{I}_n$. Show that for every vector $\mathbf{v} \in \mathbb{R}^n$, $\operatorname{Rep}_{\mathcal{B}}(\mathbf{v}) = \mathbf{A}^\top\mathbf{v}$, where $\mathcal{B}$ is the orthonormal basis formed by the columns of $\mathbf{A}$.

5. Let
$$\mathbf{A} = \begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 5 & 4\\ 1 & 3 & 4 & 5\\ 1 & 3 & 6 & 6\end{bmatrix}.$$
Use the definition (Four.I.2.1 of the textbook) to find $\det(\mathbf{A})$.

6. Find the determinant of the (row) elementary matrix of each type. Show that $\det(\mathbf{E}^{-1}) = \det(\mathbf{E})^{-1}$ if E is an elementary matrix.

7. By Gaussian elimination, if a matrix $\mathbf{A}$ has the reduced echelon form $\mathbf{R}$, then we have $\mathbf{E}_k \cdots \mathbf{E}_1\mathbf{A} = \mathbf{R}$ or $\mathbf{A} = \mathbf{E}_1^{-1} \cdots \mathbf{E}_k^{-1}\mathbf{R}$, where $\mathbf{E}_i$'s are elementary matrices. Check that
$$\det(\mathbf{A}) = \det(\mathbf{E}_1^{-1}) \cdots \det(\mathbf{E}_k^{-1}) \det(\mathbf{R}).$$
Use this to show that $\det(\mathbf{A}) \neq 0$ if and only if $\mathbf{A}$ is invertible.