# Solutions
# Sample ~~Questions~~ for Sample Questions 6

## 1. Gram-Schmidt Algorithm

$$\begin{cases} \vec{u}_1 = \vec{v}_1 \\ \vec{u}_2 = \vec{v}_2 - \text{proj}_{[\vec{u}_1]}(\vec{v}_2) \\ \vec{u}_3 = \vec{v}_3 - \text{proj}_{[\vec{u}_1]}(\vec{v}_3) - \text{proj}_{[\vec{u}_2]}(\vec{v}_3). \end{cases}$$

Recall that $\text{proj}_{[\vec{u}]}(\vec{v}) = \dfrac{\langle \vec{v}, \vec{u} \rangle}{|\vec{u}|^2} \cdot \vec{u}$.

So $\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle}{\left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right|^2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

[You may now check $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$.]

$$\vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1/3}{6/9} \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/6 \\ 2/6 \\ 2/6 \end{pmatrix} - \begin{pmatrix} -2/6 \\ 1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$$

[You may check $\langle \vec{u}_1, \vec{u}_3 \rangle = \langle \vec{u}_2, \vec{u}_3 \rangle = 0$.]

2. Normalize : $\hat{u} = \frac{1}{|\vec{u}|} \cdot \vec{u}$

So $\hat{u}_1 = \frac{1}{|\vec{u}_1|} \cdot \vec{u}_1 = \frac{1}{\sqrt{3}} \vec{u}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \rightarrow \vec{u}_1 = \sqrt{3}\,\hat{u}_1$

$\hat{u}_2 = \frac{1}{|\vec{u}_2|} \cdot \vec{u}_2 = \frac{1}{\sqrt{2/3}} \vec{u}_2 = \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \rightarrow \vec{u}_2 = \sqrt{\frac{2}{3}}\,\hat{u}_2$

$\hat{u}_3 = \frac{1}{|\vec{u}_3|} \cdot \vec{u}_3 = \frac{1}{\sqrt{1/2}} \vec{u}_3 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \rightarrow \vec{u}_3 = \frac{1}{\sqrt{2}}\,\hat{u}_3$

Exam the calculation:

$\vec{u}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \vec{u}_1 = \sqrt{3}\,\hat{u}_1 + 0\,\hat{u}_2 + 0\,\hat{u}_3$

$\vec{u}_2 = \vec{v}_2 - \frac{2}{3}\vec{u}_1 \Rightarrow \vec{v}_2 = \frac{2}{3}\vec{u}_1 + \vec{u}_2 + 0\,\vec{u}_3$

$= \frac{2}{\sqrt{3}}\hat{u}_1 + \sqrt{\frac{2}{3}}\,\hat{u}_2 + 0\,\hat{u}_3$

$\vec{u}_3 = \vec{v}_3 - \frac{1}{3}\vec{u}_1 - \frac{1}{2}\vec{u}_2 \Rightarrow \vec{v}_3 = \frac{1}{3}\vec{u}_1 + \frac{1}{2}\vec{u}_2 + \vec{u}_3$

$= \frac{1}{\sqrt{3}}\hat{u}_1 + \frac{1}{\sqrt{6}}\hat{u}_2 + \frac{1}{\sqrt{2}}\hat{u}_3$ .

So $A = Q \cdot R$

$$\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

↑ orthogonal

since $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ is orthonormal

↑ upper triangular.

3. Find a basis for $V$ first.

$V$ is the null space of $[1 \ 1 \ 1]$

So $V = \text{span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_2} \right\}$

找零解.

Apply Gram-Schmidt,

$$\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{[\vec{u}_1]}(\vec{v}_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Then $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis.

[ If you want orthonormal basis, then normalize $\vec{u}_1$ and $\vec{u}_2$ ].

[ Of course you can cheat in this special case.

That is, take $\vec{u}_2$ and $\vec{u}_3$ as in Problem 1.]

4.

Method 1.

Let $A = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix}$. Then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis.

In this case, $\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} \langle \vec{v}, \vec{u}_1 \rangle \\ \vdots \\ \langle \vec{v}, \vec{u}_n \rangle \end{pmatrix}$

$$= \begin{pmatrix} — & \vec{u}_1 & — \\ & \vdots & \\ — & \vec{u}_n & — \end{pmatrix} \begin{pmatrix} | \\ \vec{v} \\ | \end{pmatrix} = A^T \vec{v}.$$

Method 2.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \iff \vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$$

$$\iff \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{v}.$$

So, $A \cdot \mathrm{Rep}_B(\vec{v}) = \vec{v}$

Since $AA^T = A^TA = I \Rightarrow A^T = A^{-1}$.

$\Rightarrow \mathrm{Rep}_B(\vec{v}) = A^{-1}\vec{v} = A^T\vec{v}$.

5.

$$A \xrightarrow[-r_1+r_4]{\substack{-r_1+r_2\\-r_1+r_3}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix} \xrightarrow{-r_3+r_4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & 2 & \\ & 1 & 1 & 1 \\ & & 2 & 1 \end{pmatrix}$$

$\times 1$ $\qquad$ $\times 1$

$\det = -2 \leftarrow$ $\det = -2$ $\qquad$ $\det = -2 \leftarrow$

$$\xrightarrow{-r_2+r_4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & 2 & \\ & 1 & 1 & 1 \\ & & & 1 \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_4} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 1 & 1 & 1 \\ & & 2 & \\ & & & 1 \end{pmatrix}$$

$\times 1$ $\qquad$ $\times(-1)$

$\det = -2$ $\qquad$ $\det = 2 \leftarrow$

$$\xrightarrow{\frac{1}{2} r_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 1 & 1 & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix} \xrightarrow{\text{lots of row operation}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\times \frac{1}{2}$ $\qquad$ $\times 1$

$\leftarrow$ $\qquad$ $\det = 1$ $\qquad$ $\det = 1$

So $\det(A) = -2$.

6. Type 1. $r_i \leftrightarrow r_j$

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 \\ & & \vdots & \ddots & \vdots \\ & & 1 & \cdots & 0 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{matrix}, \quad E \xrightarrow{r_i \leftrightarrow r_j} I_n$$

$\uparrow$ $i$ $\quad$ $\uparrow$ $j$

So $\det(E) = -\det(I_n) = -1$.

$E^{-1} = E \Rightarrow \det(E^{-1}) = -1 = \det(E)^{-1}$

Type 2. $k r_i$

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & k & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \leftarrow i, \quad E \xrightarrow{\frac{1}{k} r_i} I_n$$

$\uparrow$ $i$

So $\frac{1}{k}\det(E) = \det(I_n) = 1 \Rightarrow \det(E) = k$.

$E^{-1}$ is the elementary matrix from $\frac{1}{k} r_i$, so $\det(E^{-1}) = \frac{1}{k} = \det(E)^{-1}$

Type 3. $k r_i + r_j$

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & k & \cdots & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{matrix}, \quad E \xrightarrow{-k r_i + r_j} I_n.$$

So $\det(E) = \det(I_n) = 1$.

$E^{-1}$ is the elementary matrix from $-k r_i + r_j$,

so $\det(E^{-1}) = 1 = \det(E)$.

7. By recording the elimination process, we can find

$$E_k \cdots E_2 E_1 A = R \quad \text{or}$$

$$A = E_1^{-1} \cdots E_k^{-1} R .$$

By observations in Problem 6,

$$\det(E_k) \cdots \det(E_1) \cdot \det(A) = \det(R)$$

$$\Updownarrow$$

$$\det(A) = \det(E_k)^{-1} \cdots \det(E_1)^{-1} \det(R)$$

$$= \det(E_1^{-1}) \cdots \det(E_k^{-1}) \cdot \det(R).$$

Note that $\det(E_1), \ldots, \det(E_k)$ are not zero, since the determinant of an elementary matrix is never zero by Problem 6.

$\det(A) \neq 0$ if and only if $\det(R) \neq 0$

if and only if ~~A is Inver~~ R has no zero rows

if and only if $R = I_n$

if and only if $A$ is invertible.