Sample Questions 7

1. Let $\mathbf{A}$ be a square matrix whose rows are $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$. Suppose $\mathbf{r}_j = \sum_{i \neq j} c_i \mathbf{r}_i$ for some j. That is, $\mathbf{r}_j$ is a linear combination of the other rows (and thus the rows form a dependent set). Show that $\det(\mathbf{A}) = 0$.

2. Let a, b, c, d be four distince real numbers and
$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}.$$
A matrix of this form is called a Vandemonde matrix. Show that $\det(\mathbf{A}) = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$. Therefore, such a matrix is invertible if a, b, c, d are distinct. (See Problem 4 in SQ5 for its applications.)

3. Write down all the $4! = 24$ different 4-permutations and their permutation matrices. Then find the determinant of each of the permutation matrices.

4. Find a formula of $\det(\mathbf{A})$ when $\mathbf{A}$ is a 4×4 matrix.

5. Let $\phi = (2, 3, 4, 5, 1)$. Find $\mathbf{P}_\phi$, $\mathbf{P}_{\phi^{-1}}$, $\mathbf{P}_\phi^\top$, and their determinants. (Try some other ϕ to convince yourself that $\det(\mathbf{P}_\phi) = \det(\mathbf{P}_\phi^\top)$.)

6. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Show that $\mathbf{AB}$ is singular when $\mathbf{A}$ is singular. Therefore,
$$\det(\mathbf{AB}) = 0 = \det(\mathbf{A}) \det(\mathbf{B})$$
when $\mathbf{A}$ is singular.

7. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Use the previous problem and Problem 7 of SQ6 to show that
$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}).$$
(Consider two cases: Whether $\mathbf{A}$ is singular or not.)