Sample Solutions for Sample Questions 7

1. Equivalently, the condition says

$$\vec{r}_j - \sum_{i \neq j} c_i \vec{r}_i = \vec{0}.$$

Therefore, $A \xrightarrow[\text{for all } i \neq j]{-c_i\vec{r}_i + \vec{r}_j} B$

and the j-th row of B is zero.

$\Rightarrow \det(B) = 0.$

Since $\xrightarrow{-c_i\vec{r}_i + \vec{r}_j}$ does not change the determinant,

$$\det(A) = \det(B) = 0.$$

2.

$$\det(A) = \det\begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{pmatrix}$$

$$= (b-a)(c-a)(d-a)\cdot\det\begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ab+a^2 \\ 0 & 1 & c+a & c^2+ac+a^2 \\ 0 & 1 & d+a & d^2+ad+a^2 \end{pmatrix}$$

$$= (b-a)(c-a)(d-a)\cdot\det\begin{pmatrix} 1 & a & a^2 & a^3 \\ & 1 & b-a & b^2+ab+a^2 \\ & 0 & c-b & c^2-b^2+ac-ab \\ & 0 & d-b & d^2-b^2+ad-ab \end{pmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\cdot\det\begin{pmatrix} 1 & a & a^2 & a^3 \\ & 1 & b-a & b^2+ab+a^2 \\ & & 1 & c+b+a \\ & & 1 & d+b+a \end{pmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\cdot\det\begin{pmatrix} 1 & a & a^2 & a^3 \\ & 1 & b-a & b^2+ab+a^2 \\ & & 1 & c+b+a \\ & & 0 & d-c \end{pmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c).$$

3.

ϕ	$(1,2,3,4)$	$(1,2,4,3)$	$(1,3,2,4)$	$(1,3,4,2)$	$(1,4,2,3)$	$(1,4,3,2)$
P_ϕ	$\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$	$\begin{pmatrix}1&&&\\&1&&\\&&&1\\&&1&\end{pmatrix}$	$\begin{pmatrix}1&&&\\&&1&\\&1&&\\&&&1\end{pmatrix}$	$\begin{pmatrix}1&&&\\&&1&\\&&&1\\&1&&\end{pmatrix}$	$\begin{pmatrix}1&&&\\&&&1\\&1&&\\&&1&\end{pmatrix}$	$\begin{pmatrix}1&&&\\&&&1\\&&1&\\&1&&\end{pmatrix}$
det	1	-1	-1	1	1	-1

ϕ	$(2,1,3,4)$	$(2,1,4,3)$	$(2,3,1,4)$	$(2,3,4,1)$	$(2,4,1,3)$	$(2,4,3,1)$
P_ϕ	$\begin{pmatrix}&1&&\\1&&&\\&&1&\\&&&1\end{pmatrix}$	$\begin{pmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{pmatrix}$	$\begin{pmatrix}&1&&\\&&1&\\1&&&\\&&&1\end{pmatrix}$	$\begin{pmatrix}&1&&\\&&1&\\&&&1\\1&&&\end{pmatrix}$	$\begin{pmatrix}&1&&\\&&&1\\1&&&\\&&1&\end{pmatrix}$	$\begin{pmatrix}&1&&\\&&&1\\&&1&\\1&&&\end{pmatrix}$
det	-1	1	1	-1	-1	1

ϕ	$(3,1,2,4)$	$(3,1,4,2)$	$(3,2,1,4)$	$(3,2,4,1)$	$(3,4,1,2)$	$(3,4,2,1)$
P_ϕ	$\begin{pmatrix}&&1&\\1&&&\\&1&&\\&&&1\end{pmatrix}$	$\begin{pmatrix}&&1&\\1&&&\\&&&1\\&1&&\end{pmatrix}$	$\begin{pmatrix}&&1&\\&1&&\\1&&&\\&&&1\end{pmatrix}$	$\begin{pmatrix}&&1&\\&1&&\\&&&1\\1&&&\end{pmatrix}$	$\begin{pmatrix}&&1&\\&&&1\\1&&&\\&1&&\end{pmatrix}$	$\begin{pmatrix}&&1&\\&&&1\\&1&&\\1&&&\end{pmatrix}$
det	1	-1	-1	1	1	-1

ϕ	$(4,1,2,3)$	$(4,1,3,2)$	$(4,2,1,3)$	$(4,2,3,1)$	$(4,3,1,2)$	$(4,3,2,1)$
P_ϕ	$\begin{pmatrix}&&&1\\1&&&\\&1&&\\&&1&\end{pmatrix}$	$\begin{pmatrix}&&&1\\1&&&\\&&1&\\&1&&\end{pmatrix}$	$\begin{pmatrix}&&&1\\&1&&\\1&&&\\&&1&\end{pmatrix}$	$\begin{pmatrix}&&&1\\&1&&\\&&1&\\1&&&\end{pmatrix}$	$\begin{pmatrix}&&&1\\&&1&\\1&&&\\&1&&\end{pmatrix}$	$\begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix}$
det	-1	1	1	-1	-1	1

Note: Always, there are $\frac{n!}{2}$ 1's and $\frac{n!}{2}$ -1's.

4.

$$\det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}.$$

$$\begin{aligned}
&afkp - aflo - agjp + agln + ahjo - ahkn \\
= \; &- bekp + belo + bgip - bglm - bhio + bhkm \\
&+ cejp - celn - cfip + cflm + chin - chjm \\
&- dejo + dekn + dfio - dfkm - dgin + dgjm.
\end{aligned}$$

5. $\phi = (2,3,4,5,1) \qquad \phi^{-1} = (5,1,2,3,4)$

$$P_\phi = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & & & & \end{pmatrix} \qquad P_{\phi^{-1}} = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{pmatrix} = P_\phi^T$$

$$\det(P_\phi) = \det(P_\phi^T) = 1.$$

6. Suppose A is singular.

$\Rightarrow \operatorname{rank}(A) < n$

$\Rightarrow \operatorname{Colspace}(A) \subsetneq \mathbb{R}^n$

← a subset, but not ~~equal~~ equal.

Observe that

$$\operatorname{Colspace}(AB) = \{ A\underline{\underline{Bx}} \mid x \in \mathbb{R}^n \}$$

← 看成一個向量

$$\subseteq \{ Ax \mid x \in \mathbb{R}^n \} = \operatorname{Colspace}(A) \subsetneq \mathbb{R}^n.$$

$\Rightarrow \operatorname{rank}(AB) < n$

$\Rightarrow AB$ is singular.

Thus, A is singular $\Rightarrow$ AB is singular.

Therefore $\det(A) = 0 \Rightarrow \det(AB) = 0$.

When A is singular,

$$\det(AB) = 0 = \det(A) \cdot \det(B).$$

7.

By Problem 6,

$$\det(AB) = \det(A)\det(B) = 0 \text{ when } A \text{ is singular.}$$

Suppose A is nonsingular,

Then the reduced echelon form of A is I,

and there are elementary matrices $E_1, \ldots, E_k$

so that $E_k \cdots E_1 A = I \Leftrightarrow A = E_1^{-1} \cdots E_k^{-1}$

and $\det(A) = \det(E_1^{-1}) \cdot \ldots \cdot \det(E_k^{-1})$

also elementary matrices.

Thus

$$AB = E_1^{-1} \cdot \ldots \cdot E_k^{-1} \cdot B.$$

This means AB is obtained from B by applying row operations corresponding to $E_k^{-1}, \ldots, E_1^{-1}$.

By definition of determinant,

$$\det(AB) = \det(E_1^{-1}) \cdots \det(E_k^{-1}) \cdot \det(B)$$

$$= \det(A) \cdot \det(B).$$