Sample Questions 8

Let $\mathbf{I}_n$ be the $n \times n$ identity matrix. Let $\mathbf{J}_n$ be the $n \times n$ all-ones matrix. Also, $\mathbf{1}$ is the all-ones vector and $\mathbf{0}$ is the zero vector.

1. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that $\det(-\mathbf{A}) = (-1)^n \det(\mathbf{A})$. Furthermore, a matrix is called *skew-symmetric* if $\mathbf{A}^\top = -\mathbf{A}$. Show that an $n \times n$ skew-symmetric matrix is always singular when n is odd.

2. Suppose $\mathbf{A}$ is an $n \times n$ orthogonal matrix. That is $\mathbf{A}\mathbf{A}^\top = \mathbf{A}^\top\mathbf{A} = \mathbf{I}_n$. Show that $|\det(\mathbf{A})| = 1$. Next, suppose $\mathbf{B}$ is a matrix whose rows $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are mutually orthogonal. Show that

   $$|\det(\mathbf{B})| = |\mathbf{v}_1| \cdots |\mathbf{v}_2|.$$

   (This is also the expected volume, the product of the length of each sides.)

3. Let R be the rectangle defined by $1 \leqslant x \leqslant 4$ and $2 \leqslant y \leqslant 4$. Define a homomorphism $t : \mathbb{R}^2 \to \mathbb{R}^2$ by $t(\mathbf{v}) = \mathbf{A}\mathbf{v}$ with $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$. Draw the region $t(R)$ and compute its area.

4. Find

   $$\det \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

   by Laplace expansion.

5. Suppose $\mathbf{A}$ is a matrix such that $\mathbf{A}\mathbf{1} = \mathbf{0}$. Show that

   $$\det(\mathbf{A}(1,1)) = -\det(\mathbf{A}(1,2)).$$

   Recall that $\mathbf{A}(i,j)$ is the matrix obtained from $\mathbf{A}$ by removing the i-th row and the j-th column. (In fact, when i is fixed, $|\det(\mathbf{A}(i,j))|$ is a constant for all j.)

6. Let $\mathbf{A} = \mathbf{I}_2$, $\mathbf{B} = \mathbf{J}_2$, and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Let

   $$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}.$$

   Find $\det(\mathbf{X})$ by the Schur complement of $\mathbf{A}$.

7. Find $\det(\mathbf{J}_n - \mathbf{I}_n)$ as a formula in n.