Sample Solutions for Sample Questions 8.

1. Let $A = \begin{pmatrix} \text{—} \ \vec{r}_1 \ \text{—} \\ \vdots \\ \text{—} \ \vec{r}_n \ \text{—} \end{pmatrix}$.

Then $-A = \begin{pmatrix} \text{—} \ -\vec{r}_1 \ \text{—} \\ \vdots \\ \text{—} \ -\vec{r}_n \ \text{—} \end{pmatrix}$.

Thus,
$$\det(-A) = (-1)\cdot \det\begin{pmatrix} \text{—} \ \vec{r}_1 \ \text{—} \\ \text{—} \ -\vec{r}_2 \ \text{—} \\ \vdots \\ \text{—} \ -\vec{r}_n \ \text{—} \end{pmatrix}$$
$$= (-1)^2 \cdot \det\begin{pmatrix} \text{—} \ \vec{r}_1 \ \text{—} \\ \text{—} \ \vec{r}_2 \ \text{—} \\ \text{—} \ -\vec{r}_3 \ \text{—} \\ \vdots \\ \text{—} \ -\vec{r}_n \ \text{—} \end{pmatrix}$$
$$= (-1)^n \cdot \det(A).$$

Suppose A is an $n\times n$ skew-symmetric matrix.

Then $A^T = -A$.

When n is odd,
$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$
$$= -\det(A).$$

$\Rightarrow \det(A) = 0$.

$\Rightarrow A$ is singular.

2. Suppose A is an $n \times n$ orthogonal matrix.

Then $AA^T = I_n$. $\Rightarrow \det(A) \cdot \det(A^T) = \det(I_n) = 1$.

~~Since A~~ $\Rightarrow \det(A) \cdot \det(A) = 1$

$\Rightarrow \det(A) = \pm 1$

and $|\det(A)| = 1$.

Suppose $B = \begin{pmatrix} \text{---} \ \vec{v}_1 \ \text{---} \\ \vdots \\ \text{---} \ \vec{v}_n \ \text{---} \end{pmatrix}$ such that

the rows are mutually orthogonal.

① If some of the rows are zero,

then

$$\det(B) = 0 = |\vec{v}_1| \cdots |\vec{v}_n|.$$

② If each row is nonzero,

write $\vec{u}_i = \dfrac{\vec{v}_i}{|\vec{v}_i|}$ and $\vec{v}_i = |\vec{v}_i| \cdot \vec{u}_i$.

Thus, $\det(B) = \det \begin{pmatrix} \text{---} \ |\vec{v}_1| \cdot \vec{u}_1 \ \text{---} \\ \vdots \\ \text{---} \ |\vec{v}_n| \vec{u}_n \ \text{---} \end{pmatrix}$

$= |\vec{v}_1| \cdots |\vec{v}_n| \cdot \underbrace{\det \begin{pmatrix} \text{---} \ \vec{u}_1 \ \text{---} \\ \vdots \\ \text{---} \ \vec{u}_n \ \text{---} \end{pmatrix}}_{\det = \pm 1 \text{ since orthogonal.}}$

$\Rightarrow |\det(B)| = |\vec{v}_1| \cdots |\vec{v}_n|$.

3.

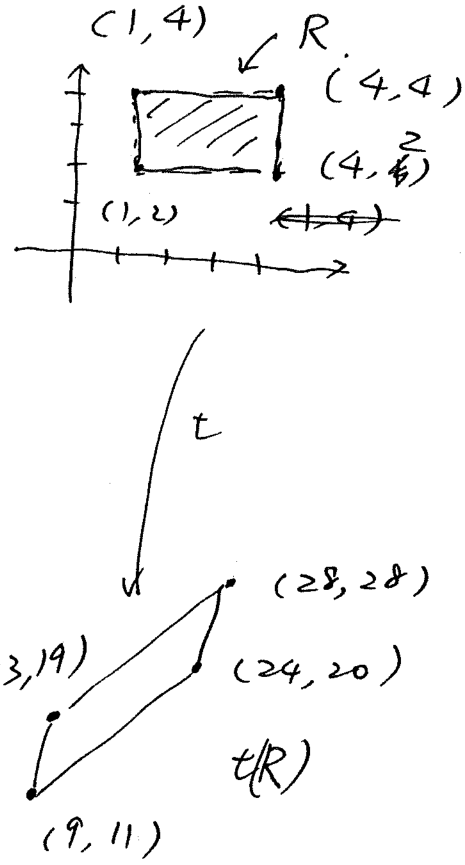
Compute

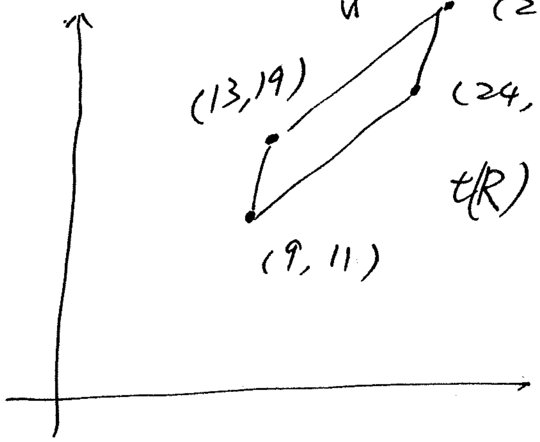
$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \cancel{\begin{pmatrix} 22 \\ 16 \end{pmatrix}} \begin{pmatrix} 24 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 28 \\ 28 \end{pmatrix}$$





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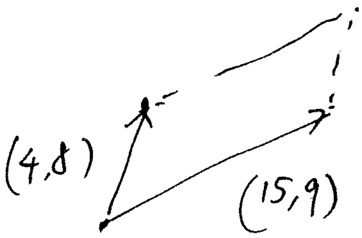

compute $\begin{vmatrix} 15 & 9 \\ 4 & 8 \end{vmatrix} = 120 - 36$

$= 84.$

area $= 84.$

Alternatively, the area of $R = 3 \cdot 2 = 6.$

area of $t(R) = \left| \det(A) \cdot 6 \right|$

$= 14 \cdot 6 = 84.$

4.

$$\det\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} = 2\cdot\det\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix} + [-(-1)]\det\begin{pmatrix} -1 & -1 & & \\ & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

$$= 4\cdot\det\begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} + 2\cdot\det\begin{pmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{pmatrix} - \det\begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

$$= 3\cdot\det\begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} + 2\cdot\det\begin{pmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

now you may use 3×3 determinant formula

$$= 3\cdot(2\cdot2\cdot2 - 2 - 2) + 2\cdot\cancel{\text{de}}(-4+1)$$

$$= 12 - 6$$

$$= 6.$$

5. Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$ ← $n \times n$ matrix.

If $A \cdot \vec{1} = \vec{0} \Rightarrow \vec{v}_1 + \cdots + \vec{v}_n = \vec{0}$.

Let $\vec{u}_i$ be a vector in $\mathbb{R}^{n-1}$ obtained from $\vec{v}_i$ by removing the first ~~entry~~ entry.

Then

$$A(1,1) = \begin{pmatrix} | & | & & | \\ \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix} \leftarrow (n-1)\times(n-1)$$

$$A(1,2) = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_3 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix}.$$

Also, $\vec{u}_1 + \cdots + \vec{u}_n = \vec{0} \Leftrightarrow \vec{u}_1 = -(\vec{u}_2 + \cdots + \vec{u}_n)$.

加過去.

Thus,

$$\det(A(1,1)) = \det\begin{pmatrix} | & | & & | \\ \vec{u}_2 & \vec{u}_3 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix}$$

$$= \det\begin{pmatrix} | & | & & | \\ (\vec{u}_2 + \cdots + \vec{u}_n) & \vec{u}_3 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix}$$

$$= \det\begin{pmatrix} | & | & & | \\ -\vec{u}_1 & \vec{u}_3 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix} = -\det(A(1,2)).$$

6. $$X = \left(\begin{array}{cc|cc} 1 & & 1 & 1 \\ & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right).$$

$$BA^{-1}B\ \cancel{X/A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$X/A = C - BA^{-1}B$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \det(X) = \det(A) \cdot \det(X/A)$$

$$= 1 \cdot (-1) = -1.$$

$$J_n - I_n = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

~~det (J~~

$$\det(J_n - I_n) = \det\begin{pmatrix} n-1 & & & \\ n-1 & 0 & & 1 \\ \vdots & & \ddots & \\ n-1 & 1 & & 0 \end{pmatrix}$$

(所有 column 加到第一個 column)

$$= (n-1) \cdot \det\begin{pmatrix} 1 & & & \\ 1 & 0 & & 1 \\ \vdots & & \ddots & \\ 1 & 1 & & 0 \end{pmatrix}$$

(提出 column 1 的 n-1)

$$= \cancel{\det} (n-1) \cdot \det\begin{pmatrix} 1 & & & \\ 1 & -1 & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & -1 \end{pmatrix}$$

(把 -1* column 1 加到每個 column)

$$= (-1)^{(n-1)} \cdot (n-1).$$

(下三角矩陣)