# Sample Questions 9

1. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices. Let $t$ be a variable. Show that $p(t) = \det(\mathbf{A} + t\mathbf{B})$ is a polynomial in $t$ with degree at most $n$. Moreover, if $\mathbf{B}$ is the identity matrix, show that $p(t)$ is a polynomial of degree $= n$.

2. Compute the adjugate for each of
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$
Then compute their inverses, if invertible.

3. Let $\mathbf{A}$ be an $n \times n$ matrix and $A_{i,j}$ the $i,j$-cofactor of $\mathbf{A}$. Suppose $\mathbf{B}$ is the matrix obtained from $\mathbf{A}$ by replacing the $k$-th row with the vector $\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$. Show that
$$\det(\mathbf{B}) = c_1 A_{i,1} + \cdots + c_n A_{i,n}.$$

4. Let $\mathbf{P}_n$ be the matrix whose $i,j$-entry is 1 when $|i-j| = 1$ and 0 otherwise. For example,
$$\mathbf{P}_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$
Find $\det(\mathbf{P}_n)$ as a formula of $n$. When $n$ is even, find the $1,1$-entry of $\mathbf{P}_n^{-1}$.

5. Let
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$
Find $\operatorname{cof}(\mathbf{A})$ and $\det(\mathbf{A})$.

6. Let $\mathbf{A}$ be as in Problem 5. Let $\mathbf{1} \in \mathbb{R}^4$ be the all-ones vector. Use Cramer's rule (and the cofactors you computed in Problem 5) to solve $\mathbf{A}\mathbf{x} = \mathbf{1}$.

7. Let $\mathbf{A}$ be an $n \times n$ matrix and $\mathbf{J}$ the $n \times n$ all-ones matrix. Show that
$$\det(\mathbf{A} + k \cdot \mathbf{J}_n) = \det(\mathbf{A}) + k \cdot \operatorname{cof}(\mathbf{A}).$$