Sample Solutions for Sample Questions 9.

1. Suppose
$$A=\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \text{ and } B=\begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix}.$$

Then
$$A+tB=\begin{pmatrix} a_{1,1}+tb_{1,1} & \cdots & a_{1,n}+tb_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1}+tb_{n,1} & \cdots & a_{n,n}+tb_{n,n} \end{pmatrix}.$$

Let $c_{i,j}=a_{i,j}+tb_{i,j}$ be the i,j-entry of $A+tB$.

Thus, $\det(A+tB)=\sum_{\phi:\text{n-permutation}} c_{1,\phi(1)}\cdot\cdots\cdot c_{n,\phi(n)}\cdot\det(P_\phi)$.

Now for each ϕ, $c_{1,\phi(1)}\cdot\cdots\cdot c_{n,\phi(n)}$
$$=(a_{1,\phi(1)}+tb_{1,\phi(1)})\cdot\cdots\cdot(a_{n,\phi(n)}+tb_{n,\phi(n)})$$
is a polynomial in t with degree at most n.

Therefore, $\det(A+tB)$ is a sum of $n!$ polynomials of degree at most n.

$\Rightarrow \det(A+tB)$ is also a polynomial of degree at most n.

1. [continued]

Now suppose $B =$ identity matrix.

Examine $\det(A+tB)$ again.

① If $\phi = (1\ 2\ \ldots\ n)$,
identity map.

then $c_{1,\phi(1)} \cdots c_{n,\phi(n)} = (a_{1,1}+t)(a_{2,2}+t) \cdots (a_{n,n}+t)$

is a polynomial of degree n.

exactly $= n$, not $\leq n$.

② If $\phi \neq$ identity map,

at least on number k has the property $\phi(k) \neq k$.

$\Rightarrow c_{1,\phi(1)} \cdots c_{n,\phi(n)} = (a_{1,\phi(1)} + tb_{1,\phi(1)}) \cdots (a_{k,\phi(k)} + t\, b_{k,\phi(k)})$
$\cdots (a_{n,\phi(n)} + tb_{n,\phi(n)})$

$b_{k,\phi(k)} = 0$.
since $B =$ identity matrix
and $\phi(k) \neq k$.

is a polynomial of degree $< n$.

$\Rightarrow \det(A+tB) =$ a polynomial of degree $= n$

+ $(n!-1)$ polynomials of degree $< n$

is a polynomial of degree $= n$.

2.

~~adj(A) =~~

For A, $[i,j\text{-cofactor}] = \begin{pmatrix} +(-3) & -(-6) & +(-3) \\ -(-6) & +(-12) & -(-6) \\ +(-3) & -(-6) & +(-3) \end{pmatrix} = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}$

$\Rightarrow \text{adj}(A) = [i,j\text{-cofactor}]^T = \begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}$.

A is not invertible since $\det(A) = 0$.

(Note that A is not invertible also means adj(A) not invertible.)

For B, $[i,j\text{-cofactor}] = \begin{pmatrix} +(-1) & -(-1) & +(1) \\ -(-1) & +(-1) & -(-1) \\ +(1) & -(-1) & +(-1) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

$\Rightarrow$ ~~adj(A)~~ $\text{adj}(B) = [i,j\text{-cofactor}]^T = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$\det(B) = 2$.

$\Rightarrow B^{-1} = \frac{1}{2}\,\text{adj}(B) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

注意這裡兩個矩陣剛好都對稱，
所以 adj(A) = [i,j-cofactor] 剛好一樣
一般矩陣記得取轉置 $\text{adj}(A) = [i,j\text{-cofactor}]^T$.

3.

First $B = \begin{pmatrix} \boxed{\quad A \quad} \\ c_1 \ \cdots \ c_n \\ \boxed{\quad A \quad} \end{pmatrix}$ ← k-th row.

so the i,j-cofactor of B is $A_{i,j}$ if $i=k$.

Expand B along the kth row.

$$\det(B) = c_1 A_{k,1} + \cdots + c_n A_{k,n}.$$

4. Use induction.

$$\det(P_1) = \det([0]) = 0$$

$$\det(P_2) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$$

$$\det(P_3) = \det\left(\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix}\right) = 0$$

$$\det(P_4) = \det\left(\begin{bmatrix} & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \end{bmatrix}\right) = 1.$$

Hypothesis: $\det(P_n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^{\frac{n}{2}} & \text{if } n \text{ even.} \end{cases}$

Now compute $\det(P_n) = -\det\begin{pmatrix} 1 & 1 & & & \\ 0 & 0 & \ddots & & 0 \\ \vdots & 1 & \ddots & \ddots & \\ 0 & 0 & & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}$ [expand along the first row]

$= -\det(P_{n-2})$ [Expand along the first column]

$\Rightarrow \begin{cases} \det(P_1), \det(P_3), \ldots = 0, 0, \ldots \\ \det(P_2), \det(P_4), \ldots = -1, 1, \ldots \end{cases}$ (交錯).

4 (continued).

Find the 1.1-entry of P_n^{-1}

$$= \frac{1}{\det(P_n)} \cdot [\text{1.1-entry of } \operatorname{adj}(P_n)]$$

$$= \frac{1}{\det(P_n)} \cdot [\text{1.1-cofactor of } P_n]$$

$$= \frac{1}{\det(P_n)} \cdot \det(P_{n-1})$$

If n is even, then

$\det(P_n) = (-1)^{\frac{n}{2}}$ and $\det(P_{n-1}) = 0$.

$\Rightarrow$ the 1.1-entry of P_n^{-1} is 0.

5. 直接計算 ... $\operatorname{adj}(A) = \begin{pmatrix} 4 & -6 & 0 & -2 \\ -6 & 12 & -6 & 0 \\ 0 & -6 & 12 & -6 \\ -2 & 0 & -6 & 4 \end{pmatrix}$

$\Rightarrow$ cof(A) = adj(A) 的 entry 和 $= -8$

$$\det(A) \overset{-r_2+r_1}{=} \det\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \overset{-r_3+r_2}{=} \det\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$\overset{-r_2+r_1}{=} \det\begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \overset{\text{expand along 1st row}}{=} -2 \cdot \det\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} = -12.$$

6.

In the computation of Problem 5,

$$[i,j\text{-factor}] = \begin{pmatrix} 4 & -6 & 0 & -2 \\ -6 & 12 & -6 & 0 \\ 0 & -6 & 12 & -6 \\ -2 & 0 & -6 & 4 \end{pmatrix} = (A_{ij}).$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

By Cramer's rule, [Note $\det(A) = -12$].

$$x_1 = \frac{\det\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}}{-12} = \frac{A_{1,1} + A_{2,1} + A_{3,1} + A_{4,1}}{-12}$$

$$= \frac{-4}{-12} = \frac{1}{3}$$

Similarly, $x_i = \dfrac{A_{1,i} + A_{2,i} + A_{3,i} + A_{4,i}}{-12}$

$$\Rightarrow \vec{x} = \begin{pmatrix} 1/3 \\ 0 \\ 0 \\ 1/3 \end{pmatrix}.$$

You may check $A \cdot \vec{x} = \vec{1}$.

7. Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$.

multilinear.

~~multileare~~

$$\det\begin{pmatrix} | & & | \\ \vec{v}_1+k\vec{1} & \cdots & \vec{v}_n+k\vec{1} \\ | & & | \end{pmatrix} = \text{sum of } 2^n \text{ terms}$$

(if $k\vec{1}$ appears in two or more columns, the term is zero.)

$$= \det\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} + \det\begin{pmatrix} | & | & & | \\ k\vec{1} & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} + \det\begin{pmatrix} | & | & | & & | \\ \vec{v}_1 & k\vec{1} & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{pmatrix}$$

$$+ \cdots + \det\begin{pmatrix} | & & | & | \\ \vec{v}_1 & \cdots & \vec{v}_{n-1} & k\vec{1} \\ | & & | & | \end{pmatrix}$$

$$= \det(A) + k(A_{1,1} + \cdots + A_{n,1}) + \cdots + k(A_{1,n} + \cdots + A_{n,n})$$

$$= \det(A) + k \cdot \mathrm{cof}(A).$$

Here $A_{i,j}$ is the i,j-cofactor of A.