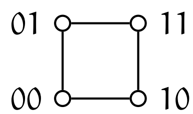
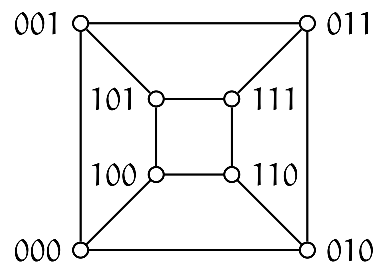
# Math589 Homework 3

**Note:** To submit the $k$-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. The *Hamming distance* between two $0,1$-strings is the number of different digits. For example, the Hamming distance between $010101$ and $111000$ is $3$. The *hypercube* $H_d$ of dimension $d$ has vertices as all $0,1$-strings of length $d$, and two vertices are adjacent if the Hamming distance of the strings is $1$. The graphs below illustrate $H_2$ and $H_3$. Show that $H_d$ is a bipartite graph for all $d$.




**Solution.** Partition $V(H_d)$ into two parts $V_{odd}$ and $V_{even}$, where $V_{odd}$ is all the $0,1$-strings in $V(H_d)$ with odd number of ones and $V_{even}$ is all the strings with even number of ones. Thus, all edges are between these two sets.

2. Let $K_n$ be the complete graph on $n$ vertices. Find the number of spanning trees on $K_n$ by the following way: Let $L_n$ be the Laplacian matrix of $K_n$. Recall that $L_n(1,1)$ is the matrix obtained from $L_n$ by removing the first row and the first column. Then the number of spanning tree equals $\det L_n(1,1)$.

   [Hint: Think about the the eigenvalues of J, the all-ones matrix.]

   **Solution.** First compute that $L_n(1,1)$ is an $(n-1) \times (n-1)$ matrix $nI_{n-1} - J_{n-1}$. The eigenvalues of $J_{n-1}$ is $\{n-1, 0^{(n-2)}\}$. Thus, the eigenvalues of $nI_{n-1} - J_{n-1}$ is $\{1, n^{(n-2)}\}$. Therefore, the determinant of $L_n(1,1)$ is $n^{n-2}$.