Math589 Homework 5

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Find a closed walk of odd length in the Kneser graph $K_{9,4}$.

   Solution. The following is a closed walk of length 9.

$$\{1,2,3,4\} \to \{5,6,7,8\}$$
$$\{1,2,3,9\} \to \{4,5,6,7\}$$
$$\{1,2,8,9\} \to \{3,4,5,6\}$$
$$\{1,7,8,9\} \to \{2,3,4,5\}$$
$$\{6,7,8,9\} \to \{1,2,3,4\}$$

2. Let S^n be the sphere of dimension n (in $\mathbb{R}^{n+1}$). That is,

$$S^n = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Consider the projection map $f : S^n \to \mathbb{R}^n$ by

$$f(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n).$$

Find a pair of antipodal points $\mathbf{x}$ and $-\mathbf{x}$ in $\mathbb{R}^{n+1}$ such that $f(\mathbf{x}) = f(-\mathbf{x})$. Are there any other pairs of the same property?

Solution. Let $\mathbf{x} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ with n entries as 0. Then

$$f(\mathbf{x}) = f(-\mathbf{x}) = (0, \ldots, 0) \in \mathbb{R}^n.$$

There are no other pairs since

$$f(\mathbf{x}) = (x_1, \ldots, x_n) = (-x_1, \ldots, x_n) = f(-\mathbf{x})$$

only when $x_1 = \cdots = x_n = 0$.