Math589 Homework 6

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Let $(X, \mathcal{O})$ be a topological space with $X = \{1, 2, 3, 4\}$ and $\mathcal{O} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$. Let $Y = \{3\}$. Find the closure $\text{cl}(Y)$, the boundary ∂Y, and the interior $\text{int}(Y)$ of Y.

   Solution. The closed sets that include Y are

   $$\{1, 2, 3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\},$$

   so $\text{cl}(Y) = \{3, 4\}$.

   Since X is the only closed set that include $X \setminus Y = \{1, 2, 4\}$, we have

   $$\text{cl}(X \setminus Y) = X.$$

   Thus, $\partial Y = \text{cl}(Y) \cap \text{cl}(X \setminus Y) = \{3, 4\} \cap X = \{3, 4\}$.

   Finally, $\text{int}(Y) = Y \setminus \partial Y = \{3, 4\} \setminus \{3, 4\} = \emptyset$.

2. Suppose X and Y are deformation retracts of Z. Show that X and Y are homotopy equivalent by Definition 1.2.2.

   Solution. Since homotopy equivalence is an equivalence relation, it is enough to show that X and Z are homotopy equivalent.

   Let $\{f_t\}_{t\in[0,1]}$ be a deformation retraction of Z onto X. By definition, $f_t : Z \to Z$ and

   - $f_0 = \mathrm{id}_Z$,
   - $f_t(x) = x$ for all $x \in X$ and for all $t \in [0, 1]$,
   - $f_1(Z) = X$.

   Define $g : X \to Z$ as the embedding map $g(x) = x \in Z$. Define $h : Z \to X$ as $h(z) = f_1(z)$. We show that $g \circ h$ is homotopic to id_Z and $h \circ g$ is homotopic to id_X. By the second condition of the deformation retration, $h \circ g$ is exactly id_X. On the other hand, the family $\{f_{1-t}\}_{t\in[0,1]}$ witnesses that $g \circ h = g \circ f_1 = f_1$ is homotopic to $f_0 = \mathrm{id}_Z$.