Math589 Homework 7

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Let $[n] = \{1, \ldots, n\}$. For any subset $\alpha \subseteq [n]$, the characteristic vector ϕ_α of α is a vector in $\mathbb{R}^n$ whose i-th entry is 1 if $i \in \alpha$ and 0 otherwise. Show that $\{\phi_\emptyset, \phi_{[1]}, \ldots, \phi_{[n]}\}$ is affinely independent.

   Solution. The set

   $$\{\phi_{[1]} - \phi_\emptyset, \ldots, \phi_{[n]} - \phi_\emptyset\} = \{\phi_{[1]}, \ldots, \phi_{[n]}\}$$

   is linearly independent.

2. Let the characteristic vectors be defined as in the previous question with $n = 3$. Let π be a permutation on $\{1, 2, 3\}$. Define a simplex

$$S_\pi = \text{conv}(\{\phi_\emptyset, \phi_{\{\pi(1)\}}, \phi_{\{\pi(1),\pi(2)\}}, \phi_{\{\pi(1),\pi(2),\pi(3)\}}\}).$$

We showed that S_π is a simplex for $\pi = \text{id}_{[3]}$. Indeed, S_π is a simplex for any permutation π. (You do not have to show this.) Show that the cubic enclosed by

$$0 \leqslant x_1, x_2, x_3 \leqslant 1$$

is the union of S_π for all permutation π.

Solution. Let (x_1, x_2, x_3) be a point in the cube. There is a permutation π such that

$$x_{\pi(1)} \geqslant x_{\pi(2)} \geqslant x_{\pi(3)}.$$

If x_1, x_2, x_3 are distinct, then this permutation is unique; otherwise, it is not. Thus, we show that this point belongs to S_π. Let's assume $\pi = \text{id}_{[3]}$ and $x_1 \geqslant x_2 \geqslant x_3$. Thus, the point belongs to S_π because

$$(x_1, x_2, x_3) = (x_1 - x_2)(1, 0, 0) + (x_2 - x_3)(1, 1, 0) + x_3(1, 1, 1) + (1 - x_1)(0, 0, 0)$$

with

$$(x_1 - x_2) + (x_2 - x_3) + x_3 + (1 - x_1) = 1.$$

The other cases are similar.