Math589 Midterm2

6 questions, 24 total points
Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [4pt] Show that the Kneser graph $K_{7,3}$ is not 2-colorable.

   Solution. It is enough to show that $K_{7,3}$ contains an odd cycle as the following.

$$\{1,2,3\} \to \{4,5,6\} \to \{1,2,7\} \to \{3,4,5\}$$
$$\{1,6,7\} \to \{2,3,4\} \to \{5,6,7\} \to \{1,2,3\}$$

2. [4pt] Let $(X, \mathcal{O})$ be a topological space with

$$X = \{1, 2, 3, 4, 5\} \text{ and } \mathcal{O} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}.$$

Let $Y = \{1\}$.

(a) Describe all closed sets on X.

(b) Find the closure $\mathrm{cl}(Y)$.

(c) Find the boundary ∂Y.

(d) Find the interior $\mathrm{int}(Y)$ of Y.

Solution. The closed sets are

$$\emptyset, X, \{2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{3, 4, 5\}.$$

The closure is $\mathrm{cl}(Y) = \{1, 3, 4, 5\}$. The boundary is $\partial Y = \{3, 4, 5\}$. And the interior is $\mathrm{int}(Y) = \{1\}$.

3. [4pt] Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, \text{ and } \mathbf{v}_4 = \begin{bmatrix} 2 \\ 5 \\ 17 \end{bmatrix}.$$

Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_3\}$ is affinely independent.

Solution. It is enough to show that $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_4 - \mathbf{v}_1\}$ is linearly independent. But this is easy since

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix}$$

is a Vandermonde matrix and is nonsingular.

4. [4pt] Let $[3] = \{1, 2, 3\}$. For any subset $\alpha \subseteq [3]$, the characteristic vector ϕ_α of α is a vector in $\mathbb{R}^3$ whose i-th entry is 1 if $i \in \alpha$ and 0 otherwise. Let π be a permutation on $\{1, 2, 3\}$. Define a simplex

$$S_\pi = \text{conv}(\{\phi_\emptyset, \phi_{\{\pi(1)\}}, \phi_{\{\pi(1),\pi(2)\}}, \phi_{\{\pi(1),\pi(2),\pi(3)\}}\}).$$

Then the cube enclosed by

$$0 \leqslant x_1, x_2, x_3 \leqslant 1$$

is the union of S_π for all permutation π. (You do not have to show this.) Let $\mathbf{v} = (0.2, 0.7, 0.3)^\top \in \mathbb{R}^3$ be a point in the cube. Which simplex S_π does $\mathbf{v}$ belongs to?

Solution. Since $0.7 > 0.3 > 0.2$, the point $\mathbf{v}$ belongs to S_π with

$$\pi(1) = 2, \pi(2) = 3, \pi(3) = 1.$$

5. [4pt] What is a simplex? What is a simplicial complex?

Solution. A simplex is the convex hull of a finite affinely independent set. A simplicial complex Δ is a collection of simplices such that:

(a) if $\sigma \in \Delta$, then any face of σ is also in Δ, and

(b) if $\sigma_1, \sigma_2 \in \Delta$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

6. [4pt] Let C_4 be the cycle on 4 vertices. Let L be the Laplacian matrix of C_4. Find the eigenvalues and an eigenbasis of L.

Solution. The Laplacian matrix is

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

By direct computation, the eigenvalues are $0, 2, 2, 4$. The the columns of

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

form an eigenbasis for L.