國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

離散數學（一） **MATH 203: Discrete Mathematics I**

第一次期中考 October 13, 2020 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Recall that $H_k^n = \binom{n+k-1}{k}$ counts the number of integer solutions of

$$x_1 + \cdots + x_n = k, \quad x_i \geq 0$$

for all $i = 1, \ldots, n$. **Use double counting** to prove that

$$H_k^{n+1} = H_k^n + H_{k-1}^n + \cdots + H_0^n.$$

Consider the equation

$$x_1 + \dots + x_n + x_{n+1} = k, \quad x_i \geq 0 \text{ for all } i = 1, \dots, n, n+1.$$

Let S be the set of integer solutions of the equation above.

Define S_j as the set of integer solutions of the equation above with $x_{n+1} = j$.

Thus, $S = S_0 \cup S_1 \cup \dots \cup S_k$.

Now $|S| = H_k^{n+1}$

$|S_j| = H_{k-j}^n$ since it is the number of integer solutions

of $x_1 + \dots + x_n + j = k, \quad x_i \geq 0$

$$\Updownarrow$$

$x_1 + \dots + x_n = k - j, \quad x_i \geq 0.$

Thus, $H_k^{n+1} = H_k^n + H_{k-1}^n + \dots + H_0^n$.

2. [5pt] **Use mathematical induction** to prove that

$$n^3 + 2n \text{ is divisible by 3 for all integer } n \geq 1.$$

Let

S_n : n^3+2n is divisible by 3

be an open statement.

Claim: S_n is true for all $n \geq 1$.

Base step:

When $n=1$, $n^3+2n = 1^3+2 = 3$ is divisible by 3.

So S_1 is True.

Inductive step:

Suppose S_k is True. [Inductive hypothesis].

Then $(k+1)^3+2(k+1) = (k^3+2k) + 3k^2+3k+1+2$

$$=(k^3+2k)+3(k^2+k+1).$$

By the inductive hypothesis,

$$3 \mid k^3+2k.$$

Therefore, $3 \mid (k+1)^3+2(k+1)$ and S_{k+1} is True.

By mathematical induction, S_n is True for all $n \geq 1$.

3. [5pt] Prove that any set $S \subseteq \{1, \dots, 140\}$ with $|S| = 71$ contains two numbers a and b such that a is divisible by b.

Define $B_i = \{ i \cdot 2^p : p = 0, 1, \dots, \ i \cdot 2^p \overset{\le 140}{\cancel{\in S}} \}$.

e.g. $B_1 = \{1, 2, 4, 8, \dots, 128\}$.

Then ~~S~~ $B_1 \cup B_3 \cup \dots \cup B_{139} = \{1, \dots, 140\}$.

Consider 71 "pigeons" and 70 "boxes" $B_1, B_3, \dots, B_{139}$.

By Pigeon Hole Principal,

at least one of B_i $(i = 1, 3, \dots, 139)$

contains two elements (or more) in S.

Say $a, b \in B_i$.

We may assume $a = i \cdot 2^p$, $b = i \cdot 2^q$ with $\cancel{p<q}$ $p > q$.

$\Rightarrow$ a is divisible by b.

4. [5pt] Let $m = 2020$ and $n = 109$. ~~$m = 2021$ and $n = 110$.~~ Find integers a and b such that

$$am + bn = 1.$$

2020	109	
1	0	2020
0	1	109
1	-18	58
-1	19	~~4~~ 51
2	-37	7
-15	278	2
47	-871	1

$2020 \div 109 = 18 \dots 58$

~~$58 \div 18 = 3 \dots 4$~~

$109 \div 58 = 1 \dots 51$

$58 \div 51 = 1 \dots 7$

$51 \div 7 = 7 \dots 2$

$7 \div 2 = 3 \dots 1$

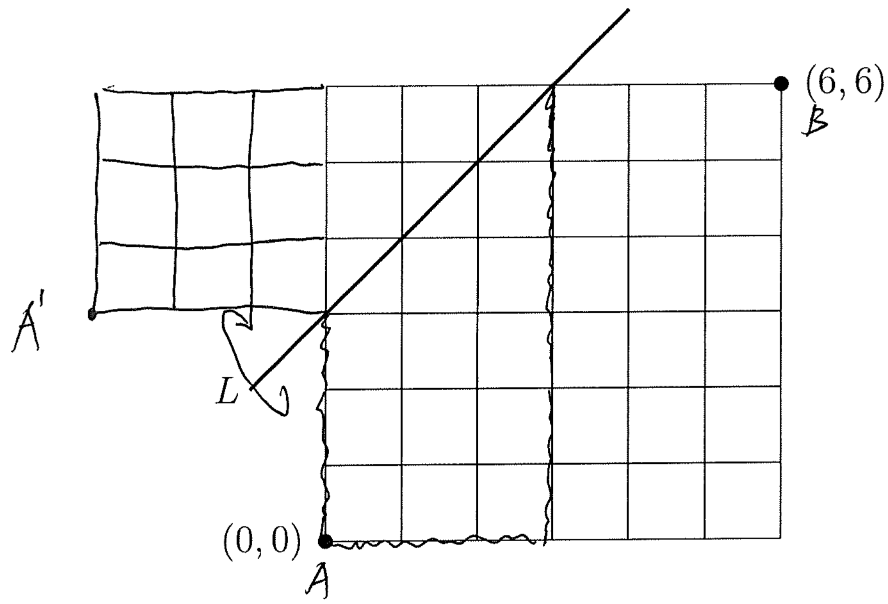
~~B~~ Double check :

$$47 \cdot 2020 + (-871) \cdot 109 = 1.$$

$\Rightarrow$ $a = 47$, $b = -871$

5. [extra 2pt] Consider two possible moves $\rightarrow$: $(1,0)$ and $\uparrow$: $(0,1)$. Count the number of ways to go from $(0,0)$ to $(6,6)$ such that

- each step is either $\rightarrow$ or $\uparrow$, and
- it **touches** the line $L : y = x + 3$.



The number of desired walks

= number of walks from A' to B.

$= \binom{3+9}{3} = \binom{12}{3} = 220$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
~~5~~	~~5~~	
~~6~~ 5.	2	
Total	~~25~~ (+2)	

20.