國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

期末考 June 8, 2020 **Final Exam**

姓名 Name : solution

學號 Student ID # :

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	~~9~~ 8 **pages** of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Give an example of a 2×2 matrix $A = [a_{ij}]$ such that A is an orthogonal matrix with $a_{12} \neq 0$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. [1pt] Give an example of a 2×2 matrix $A = [a_{ij}]$ such that A is diagonalizable and $a_{12} \neq 0$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

3. [1pt] Give an example of a 2×2 matrix $A = [a_{ij}]$ such that A is not diagonalizable and $a_{12} \neq 0$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

4. [1pt] Find a 2×2 matrix $A = [a_{ij}]$ such that the eigenvalues of A are $\{1, 5\}$ and $a_{12} = a_{21} = 2$.

$$\begin{pmatrix} a & 2 \\ 2 & b \end{pmatrix}$$

$a+b = 1+5 = 6$

$ab - 4 = 1 \cdot 5 = 5$

$\Rightarrow ab = 9$

$\Rightarrow a = b = 3.$

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

5. [1pt] Give an example of a 5×5 matrix A whose only eigenvalue is 3 with algebraic multiplicity 5 and geometric multiplicity 2.

$$\begin{pmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & 1 & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix}$$

6. Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$, the space of all 2×3 real matrices. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) [1pt] Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(M)$.

$$\mathrm{Rep}_{\mathcal{B}}(M) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \quad A \cdot \mathrm{Rep}_{\mathcal{B}}(M) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 6 \\ 5 \end{pmatrix} = \mathrm{Rep}_{\mathcal{B}}(f(M))$$

$$\Rightarrow \underline{f(M) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 6 & 5 \end{pmatrix}}$$

(b) [2pt] Find the range of f.

~~Let M~~ $\mathrm{Range}(A) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$

$\Rightarrow$ $\underline{\mathrm{range}(f) = \mathrm{span}\{E_{12}, E_{11}, E_{23}, E_{22}\}}$

(or $\left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$.)

(c) [2pt] Find the nullspace of f.

$\mathrm{nullspace}(A) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$

$\Rightarrow$ $\underline{\mathrm{null\ space}(f) = \mathrm{span}\{E_{13}, E_{21}\}}$

(or $\left\{ \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$)

7. [5pt] Let

$$A = \begin{bmatrix} 3 & \overset{8}{\cancel{17}} & \overset{-24}{\cancel{-51}} \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Find an invertible matrix Q and a diagonal matrix D such that $AQ = QD$.

$$\text{Char poly} = (3-x)(5-x)(7-x)$$

$$\Rightarrow \text{eigenvalues } \lambda = 3, 5, 7.$$

$$\lambda = 3, \quad A - 3I = \begin{pmatrix} 0 & \overset{8}{\cancel{14}} & \overset{-24}{\cancel{-51}} \\ & 2 & 6 \\ & & 4 \end{pmatrix} \Rightarrow E_\lambda = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}.$$

$$\lambda = 5, \quad A - 5I = \begin{pmatrix} -2 & \overset{8}{\cancel{12}} & \overset{-24}{\cancel{-51}}\ \cancel{-45} \\ & 0 & 6 \\ & & 2 \end{pmatrix} \Rightarrow E_\lambda = \text{span}\left\{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}\right\}$$

$$\lambda = 7, \quad A - 7I = \begin{pmatrix} -4 & 8 & -24 \\ & -2 & 6 \\ & & 0 \end{pmatrix} \Rightarrow E_\lambda = \text{span}\left\{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}\right\}.$$

$$\Rightarrow \quad \underline{\underline{Q = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}}}, \quad \underline{\underline{D = \begin{pmatrix} 3 & & \\ & 5 & \\ & & 7 \end{pmatrix}}}$$

8. [5pt] Let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$p(x) = \det(A - xI) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$$

its characteristic polynomial. Find a_0, a_1, a_2, a_5, and a_6.

$a_0 = (-1)^6 = 1$

$a_1 = (-1)^5 \cdot tr(A) = 0$

$a_2 = (-1)^4 \cdot$ sum of 2x2 principal minors $= 5 \cdot \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} +$ many $\det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= 5$

~~$a_3 =$~~

$a_5 = (-1) \cdot$ sum of 5x5 principal minors

$$= (-1) \cdot \left[\begin{vmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & 1 & 0 & -1 & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & & & & \\ & 0 & -1 & & \\ & 1 & 0 & -1 & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & 1 & 0 & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & 1 & 0 & -1 & \\ & & 1 & 0 & \\ & & & & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & 1 & 0 & -1 & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{vmatrix} \right] = 0.$$

$a_6 = \det(A) = 1.$

9. [5pt] Let J_n be the $n \times n$ all-ones matrix. Let I_n be the $n \times n$ identity matrix. Find $\det(J_n + I_n)$ as a formula of n. Make sure to justify every step of your argument.

$$\begin{vmatrix} 2 & & & 1 \\ & 2 & & \\ & & \ddots & \\ 1 & & & 2 \end{vmatrix} = \begin{vmatrix} n+1 & n+1 & \cdots & n+1 \\ & 2 & & 1 \\ & & \ddots & \\ & 1 & & 2 \end{vmatrix}$$ (全加到第一列)

$$= (n+1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & 2 & & 1 \\ & & \ddots & \\ & 1 & & 2 \end{vmatrix}$$ (提出 $n+1$)

$$= (n+1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{vmatrix}$$ (每列扣掉第一列)

$= n+1$.

10. [5pt] Let U be an $n \times n$ real upper-triangular matrix. Show that if $UU^\top = U^\top U$, then U is a diagonal matrix.

Let $U = [u_{ij}]$

Compare the 1,1-entry of $UU^T = U^TU$.

$$\Rightarrow \sum_{j=1}^{n} u_{1j}^2 = u_{11}^2 \Rightarrow u_{12} = u_{13} = \dots = u_{1n} = 0.$$

Compare the 2,2-entry of $UU^T = U^TU$.

$$\Rightarrow \sum_{j=2}^{n} u_{2j}^2 = u_{22}^2 \Rightarrow u_{23} = u_{24} = \dots = u_{2n} = 0.$$

Suppose $u_{ij} = 0$ for all $i<j$ and $i<k$.

Then compare the k,k-entry of $UU^T = U^TU$

$$\Rightarrow \sum_{j=k}^{n} u_{kj}^2 = u_{kk}^2 \Rightarrow u_{k,k+1} = \dots = u_{kn} = 0.$$

By induction, U is a diagonal matrix.

11. [extra 5pt] Let

$$A = \begin{bmatrix} -3 & 10 \\ 1 & 0 \end{bmatrix}.$$

Find A^{100}. [Hint: Write A as $\cancel{Q^{-1}DQ}$.]
QDQ^{-1}

Diagonalize A:

$$\det(A - xI) = \cancel{3}\; x^2 + 3x - 10 = (x+5)(x-2).$$

$$\lambda = 2, \quad A - 2I = \begin{pmatrix} -5 & 10 \\ 1 & -2 \end{pmatrix} \longrightarrow \text{nullspace} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}.$$

$$\lambda = -5, \quad A + 5I = \begin{pmatrix} 2 & 10 \\ 1 & 5 \end{pmatrix} \longrightarrow \text{nullspace} = \text{span}\left\{\begin{pmatrix} 5 \\ -1 \end{pmatrix}\right\}.$$

Let $Q = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & \\ & -5 \end{pmatrix}.$

Then $A = QDQ^{-1}$

$$\Rightarrow A^{100} = Q D^{100} Q^{-1}$$

$$= \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^{100} & \\ & 5^{100} \end{pmatrix} \begin{pmatrix} -1 & -5 \\ -1 & 2 \end{pmatrix} \Big/ -7$$

$$= -\frac{1}{7} \begin{pmatrix} 2 \cdot 2^{100} & 5 \cdot 5^{100} \\ 2^{100} & -1 \cdot 5^{100} \end{pmatrix} \begin{pmatrix} \cancel{2}\, -1 & -5 \\ -1 & 2 \end{pmatrix}$$

$$= -\frac{1}{7} \begin{pmatrix} -2 \cdot 2^{100} - 5 \cdot 5^{100} & -10 \cdot 2^{100} + 10 \cdot 5^{100} \\ -2^{100} + 5^{100} & -5 \cdot 2^{100} - 2 \cdot 5^{100} \end{pmatrix}.$$

12. [extra 2pt] Let

$$p(x) = x^3 + x^2 - 2x.$$

Find a 3×3 matrix A such that $p(A) = O$ and the three eigenvalues of A are all distinct.

$p(x) = x(x+2)(x-1)$

pick $A = \begin{pmatrix} 0 & & O \\ & 1 & \\ O & & -2 \end{pmatrix}$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	30 (+7)	