國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

期末考 June 8, 2020 **Final Exam**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	8 ~~9~~ **pages** of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Give an example of a 2×2 matrix $A = [a_{ij}]$ such that A is an orthogonal matrix with $a_{21} \neq 0$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. [1pt] Give an example of a 2×2 matrix $A = [a_{ij}]$ such that A is diagonalizable and $a_{21} \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

3. [1pt] Give an example of a 2×2 matrix $A = [a_{ij}]$ such that A is not diagonalizable and $a_{21} \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

4. [1pt] Find a 2×2 matrix $A = [a_{ij}]$ such that the eigenvalues of A are $\{1, 3\}$ and $a_{12} = a_{21} = 1$.

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \quad \begin{array}{l} a+b = 1+3 = 4 \\ ab \cancel{*} -1 = 1 \cdot 3 = 3 \end{array} \Rightarrow \begin{cases} a = 2 \\ b = 2 \end{cases} \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

5. [1pt] Give an example of a 5×5 matrix A whose only eigenvalue is 2 with algebraic multiplicity 5 and geometric multiplicity 3.

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{pmatrix}$$

6. Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$, the space of all 2×3 real matrices. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\text{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [1pt] Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(M)$.

$$\cancel{f}\ A \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ c \\ b \\ e \\ d \\ 0 \end{pmatrix}$$

$$f(M) = \begin{pmatrix} 0 & 3 & 2 \\ 5 & 4 & 0 \end{pmatrix}$$

So $f\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & c & b \\ e & d & 0 \end{pmatrix}$.

(b) [2pt] Find the range of f.

$$\text{range}(f) = \left\{ \begin{pmatrix} 0 & c & b \\ e & d & 0 \end{pmatrix} : b, c, d, e \in \mathbb{R} \right\}$$

or $\text{span}\{E_{12}, E_{13}, E_{21}, E_{22}\}$.

(c) [2pt] Find the nullspace of f.

$$\text{nullspace}(f) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & f \end{pmatrix} : a, f \in \mathbb{R} \right\}.$$

or $\text{span}\{E_{11}, E_{23}\}$

7. [5pt] Let

$$A = \begin{bmatrix} 2 & 10 & -20 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix}.$$

Find an invertible matrix Q and a diagonal matrix D such that $AQ = QD$.

Char poly $= (2-x)(4-x)(6-x)$.

$\rightarrow$ eigenvalues $\lambda = 2, 4, 6$.

$\lambda = 2$, $A - 2I = \begin{pmatrix} 0 & 10 & -20 \\ & 2 & 4 \\ & & 4 \end{pmatrix} \leadsto$ nullspace $= \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.

$\lambda = 4$, $A - 4I = \begin{pmatrix} -2 & 10 & -20 \\ & 0 & 4 \\ & & 2 \end{pmatrix} \leadsto$ nullspace $= \text{span}\left\{\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\lambda = 6$, $A - 6I = \begin{pmatrix} -4 & 10 & -20 \\ & -2 & 4 \\ & & 0 \end{pmatrix} \leadsto$ nullspace $= \text{span}\left\{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right\}$.

Let $\underline{\underline{Q = \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}}$, $\underline{\underline{D = \begin{pmatrix} 2 & & \\ & 4 & \\ & & 6 \end{pmatrix}}}$

Then $AQ = QD$.

8. [5pt] Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$p(x) = \det(A - xI) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$$

its characteristic polynomial. Find a_0, a_1, a_2, a_5, and a_6.

$a_0 = (-1)^6 = \underline{\underline{1}}$

$a_1 = (-1)^5 \cdot tr(A) = \underline{\underline{0}}$

$a_2 = (-1)^4 \cdot$ sum of 2x2 principal minors

$= 5 \cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \text{many} \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= \underline{\underline{-5}}$.

$a_5 = (-1)^{1} \cdot$ sum of 5x5 principal minors

$$= -\left[\begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} 0 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} \right] = \underline{\underline{0}}$$

$a_6 = \det(A) = \underline{\underline{-1}}$.

9. [5pt] Let J_n be the $n \times n$ all-ones matrix. Let I_n be the $n \times n$ identity matrix. Find $\det(J_n + I_n)$ as a formula of n. Make sure to justify every step of your argument.

See ver. A.

10. [5pt] Let U be an $n \times n$ real upper-triangular matrix. Show that if $UU^\top = U^\top U$, then U is a diagonal matrix.

See ver. A.

11. [extra 5pt] Let

$$A = \begin{bmatrix} -1 & 12 \\ 1 & 0 \end{bmatrix}.$$

Find A^{100}. [Hint: Write A as $\cancel{Q^{-1}DQ}$.]

QDQ^{-1}

Diagonalize A:

$$\det(A - xI) = x^2 + x - 12 = (x+4)(x-3).$$

$$\lambda = 3, \quad A - 3I = \begin{pmatrix} -4 & 12 \\ 1 & -3 \end{pmatrix} \rightsquigarrow \text{nullspace} = \text{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}.$$

$$\lambda = -4, \quad A + 4I = \begin{pmatrix} 3 & 12 \\ 1 & 4 \end{pmatrix} \rightsquigarrow \text{nullspace} = \text{span}\left\{\begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\}.$$

Let $Q = \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & \\ & -4 \end{pmatrix}$

Then $AQ = QD \Leftrightarrow A = QDQ^{-1}$.

$$A^{100} = QD^{100}Q^{-1}$$

$$= \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^{100} & \\ & 4^{100} \end{pmatrix} \begin{pmatrix} -1 & -4 \\ -1 & 3 \end{pmatrix} \Big/ -7$$

$$= \begin{pmatrix} 3 \cdot 3^{100} & 4 \cdot 4^{100} \\ 1 \cdot 3^{100} & -4^{100} \end{pmatrix} \begin{pmatrix} -1 & -4 \\ -1 & 3 \end{pmatrix} \Big/ -7$$

$$= -\frac{1}{7} \begin{pmatrix} -3 \cdot 3^{100} - 4 \cdot 4^{100} & -12 \cdot 3^{100} + 12 \cdot 4^{100} \\ -3^{100} + 4^{100} & -4 \cdot 3^{100} - 3 \cdot 4^{100} \end{pmatrix}$$

12. [extra 2pt] Let

$$p(x) = x^3 - x^2 - 2x.$$

Find a 3×3 matrix A such that $p(A) = O$ and the three eigenvalues of A are all distinct.

$p(x) = x(x-2)(x+1)$

Pick $A = \begin{pmatrix} 0 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	30 (+7)	