國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） MATH 104 / GEAI 1209: Linear Algebra II

第一次期中考 March 23, 2020 **Midterm 1**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**25 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ the standard basis of $\mathbb{R}^2$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be another basis of $\mathbb{R}^2$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   (a) [1pt] Find $\text{Rep}_{\mathcal{E}}(\mathbf{v})$.

   (b) [1pt] Find $\text{Rep}_{\mathcal{B}}(\mathbf{v})$.

2. Let $\mathbf{p} = x^2 + 3x + 2$ be a polynomial in $\mathcal{P}_2$, the space of all polynomials of degree at most 2.

   (a) [1pt] Let $\mathcal{B} = \{1, x, x^2\}$ be a basis of $\mathcal{P}_2$. Find $\text{Rep}_{\mathcal{B}}(\mathbf{p})$.

   (b) [1pt] Let $\mathcal{C} = \{1, x + 1, (x + 1)^2\}$ be a basis of $\mathcal{P}_2$. Find $\text{Rep}_{\mathcal{C}}(\mathbf{p})$.

   (c) [1pt] Let $\mathcal{D} = \{x^2, x, 1\}$ be a basis of $\mathcal{P}_2$. Find $\text{Rep}_{\mathcal{D}}(\mathbf{p})$.

3. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ another basis of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) [2pt] Find a matrix M such that $M \operatorname{Rep}_{\mathcal{B}}(\mathbf{v}) = \operatorname{Rep}_{\mathcal{E}}(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^3$.

(b) [3pt] Find a matrix N such that $N \operatorname{Rep}_{\mathcal{E}}(\mathbf{v}) = \operatorname{Rep}_{\mathcal{B}}(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^3$.

4. Define three polynomials as follows.

$$f_1(x) = \frac{1}{2}(x-2)(x-3)$$
$$f_2(x) = -(x-1)(x-3)$$
$$f_3(x) = \frac{1}{2}(x-1)(x-2)$$

It is known that $\mathcal{B} = \{f_1, f_2, f_3\}$ is a basis of $\mathcal{P}_2$, the space of all polynomials of degree at most 2.

(a) [2pt] Let $\mathbf{p}(x) = 3x^2 + 4x + 5$. Find $\mathrm{Rep}_{\mathcal{B}}(\mathbf{p})$.

(b) [3pt] Let $\mathcal{D} = \{1, x+2, (x+2)^2\}$ be another basis of $\mathcal{P}_2$. Find a matrix M such that $M\,\mathrm{Rep}_{\mathcal{D}}(\mathbf{q}) = \mathrm{Rep}_{\mathcal{B}}(\mathbf{q})$ for any $\mathbf{q} \in \mathcal{P}_2$.

5. [5pt] Define a map $f : \mathbb{R}^4 \to \mathbb{R}^3$ by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 3 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}.$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \text{ and.}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Find $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f)$.

6. Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$, the space of all 2×3 real matrices. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [1pt] Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(M)$.

(b) [2pt] Find the range of f.

(c) [2pt] Find the nullspace of f.

7. [extra 2pt] Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a map defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Find two bases $\mathcal{B}$ and $\mathcal{D}$ of $\mathbb{R}^3$ such that $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f)$ is the identity matrix.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	25 (+2)	