國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第一次期中考 March 23, 2020 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**25 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ the standard basis of $\mathbb{R}^2$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be another basis of $\mathbb{R}^2$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) [1pt] Find $\mathrm{Rep}_{\mathcal{E}}(\mathbf{v})$.

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{e}_1} + 5\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{e}_2} \Rightarrow \underline{\underline{Rep_{\mathcal{E}}(\vec{v}) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}}}$$

(b) [1pt] Find $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v})$.

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 4\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{v}_1} - 1\underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{v}_2} \Rightarrow \underline{\underline{Rep_{\mathcal{B}}(\vec{v}) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}}}$$

2. Let $\mathbf{p} = x^2 + 3x + 2$ be a polynomial in $\mathcal{P}_2$, the space of all polynomials of degree at most 2.

(a) [1pt] Let $\mathcal{B} = \{1, x, x^2\}$ be a basis of $\mathcal{P}_2$. Find $\mathrm{Rep}_{\mathcal{B}}(\mathbf{p})$.

$$p = 2\cdot 1 + 3\cdot x + 1\cdot x^2 \Rightarrow \underline{\underline{Rep_{\mathcal{B}}(p) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}}$$

(b) [1pt] Let $\mathcal{C} = \{1, x+1, (x+1)^2\}$ be a basis of $\mathcal{P}_2$. Find $\mathrm{Rep}_{\mathcal{C}}(\mathbf{p})$.

$$p = a + b(x+1) + c(x+1)^2$$

代入 $x = -1 \Rightarrow a = (-1)^2 + 3(-1) + 2 = 0$

$$\Rightarrow \underline{\underline{Rep_{\mathcal{C}}(p) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}}$$

$$x^2 + 3x + 2 = 0 + b(x+1) + c(x+1)^2$$

~~係~~ 比較係數 $\Rightarrow b = 1, c = 1$

(c) [1pt] Let $\mathcal{D} = \{x^2, x, 1\}$ be a basis of $\mathcal{P}_2$. Find $\mathrm{Rep}_{\mathcal{D}}(\mathbf{p})$.

$$p = 1\cdot x^2 + 3\cdot x + 2\cdot 1 \Rightarrow \underline{\underline{Rep_{\mathcal{D}}(p) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}}.$$

3. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ another basis of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) [2pt] Find a matrix M such that $M\,\mathrm{Rep}_{\mathcal{B}}(\mathbf{v}) = \mathrm{Rep}_{\mathcal{E}}(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^3$.

$$M = \mathrm{Rep}_{\mathcal{B}\,\cancel{\mathcal{B}}_{\mathcal{E}}}(id) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

(b) [3pt] Find a matrix N such that $N\,\mathrm{Rep}_{\mathcal{E}}(\mathbf{v}) = \mathrm{Rep}_{\mathcal{B}}(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^3$.

$$N = M^{-1}.$$

$$(M \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \qquad \text{So } N = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

4. Define three polynomials as follows.

$$\left.\begin{aligned} f_1(x) &= \frac{1}{2}(x-2)(x-3) \\ f_2(x) &= -(x-1)(x-3) \\ f_3(x) &= \frac{1}{2}(x-1)(x-2) \end{aligned}\right\}$$

Lagrange poly based on 1, 2, 3.

It is known that $\mathcal{B} = \{f_1, f_2, f_3\}$ is a basis of $\mathcal{P}_2$, the space of all polynomials of degree at most 2.

(a) [2pt] Let $\mathbf{p}(x) = 3x^2 + 4x + 5$. Find $\mathrm{Rep}_{\mathcal{B}}(\mathbf{p})$.

$p(1) = 12$ $\Rightarrow p = 12f_1 + 25f_2 + 44f_3$.

$p(2) = 12 + 8 + 5 = 25$

$p(3) = 27 + 12 + 5 = 44$.

$$\mathrm{Rep}_{\mathcal{B}}(p) = \begin{pmatrix} 12 \\ 25 \\ 44 \end{pmatrix}.$$

(b) [3pt] Let $\mathcal{D} = \{1, x+2, (x+2)^2\}$ be another basis of $\mathcal{P}_2$. Find a matrix M such that $M\,\mathrm{Rep}_{\mathcal{D}}(\mathbf{q}) = \mathrm{Rep}_{\mathcal{B}}(\mathbf{q})$ for any $\mathbf{q} \in \mathcal{P}_2$.

$M = \mathrm{Rep}_{D,B}(id)$

$q(x) = q(1)\cdot f_1(x) + q(2) f_2(x) + q(3) f_3(x)$

$1 = 1\cdot f_1 + 1\cdot f_2 + 1\cdot f_3$

$x+2 = 3\cdot f_1 + 4\cdot f_2 + 5\cdot f_3$ $\Rightarrow M = \begin{pmatrix} 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \\ 1 & 5 & 5^2 \end{pmatrix}$

$(x+2)^2 = 3^2\cdot f_1 + 4^2\cdot f_2 + 5^2\cdot f_3$

5. [5pt] Define a map $f : \mathbb{R}^4 \to \mathbb{R}^3$ by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 3 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}.$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \text{ and.}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Find $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f)$.

$$f(\vec{V_1}) = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = 4\vec{u_1} + 0\vec{u_2} + 0\vec{u_3}$$

$$f(\vec{V_2}) = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = 0\vec{u_1} + 6\vec{u_2} + 0\vec{u_3}$$

$$\to \underline{\underline{\mathrm{Rep}_{B,D}(f) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}}$$

$$f(\vec{V_3}) = f(\vec{V_4}) = \vec{0}$$

6. Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$, the space of all 2×3 real matrices. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [1pt] Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(M)$.

$$\mathrm{Rep}_B(M) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}. \quad A \cdot \mathrm{Rep}_B(M) = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{f(M) = \begin{pmatrix} 3 & 4 & 5 \\ 6 & 0 & 0 \end{pmatrix}}$$

(b) [2pt] Find the range of f.

$$\text{Col space}(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ 0 \\ 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

$$\Rightarrow \underline{\text{range}(f) = \left\{ \begin{pmatrix} a & b & c \\ d & 0 & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}}.$$

(c) [2pt] Find the nullspace of f.

$$\text{Nullspace}(A) = \left\{ \begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

$$\Rightarrow \underline{\text{Nullspace}(f) = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}}.$$

7. [extra 2pt] Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a map defined by $f(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Find two bases $\mathcal{B}$ and $\mathcal{D}$ of $\mathbb{R}^3$ such that $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f)$ is the identity matrix.

Let $\underline{\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}}$ be the standard basis of R^3.

Then $f(\vec{e}_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \leftarrow$ call $\vec{u}_1$

$f(\vec{e}_2) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \leftarrow \vec{u}_2$

$f(\vec{e}_3) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \leftarrow \vec{u}_3$.

Let $\underline{\mathcal{D} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}}$

Then $\mathrm{Rep}_{B,D}(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The answer is not unique.

[END]

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1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	25 (+2)	