國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第二次期中考 May 4, 2020 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **7 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } V = \operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2\}).$$

(a) [2pt] Find two vectors in $V^\perp$ such that they are independent.

Solve $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \vec{v} = \vec{0}$

$\Rightarrow \vec{v}$ can be $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ and they are indep.

(b) [1pt] Find a vector that is not in $V^\perp$.

$\vec{v}_1 \notin V^\perp$

2. Let

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{bmatrix}.$$

By the permutation expansion, $\det(A) = afkp + \cdots$ is the sum of 4! terms.

(a) [1pt] Find a term of $\det(A)$ with positive sign (other than $afkp$).

$dgjm$

(b) [1pt] Find a term of $\det(A)$ with negative sign.

$bglm$

3. Let

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 3y + 2z = 0 \right\}.$$

(a) [2pt] Find a basis of V.

~~y~~ y, z are free variables.

$y=1, z=0 \Rightarrow \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$y=0, z=1 \Rightarrow \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

basis $= \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) [3pt] Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find the projection of $\mathbf{u}$ onto the plane V.

Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

The projection of $\vec{u}$ onto Colspace(A)

$= A(A^TA)^{-1}A^T\vec{u}$

$A^TA = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 10 & -6 \\ -6 & 5 \end{pmatrix}$

$(A^TA)^{-1} = \frac{1}{14} \cdot \begin{pmatrix} 5 & 6 \\ 6 & 10 \end{pmatrix}$

$A^Tu = \begin{pmatrix} \cancel{4}\ 2 \\ -1 \end{pmatrix}$

$(A^TA)^{-1} \cdot A^Tu = \frac{1}{14} \begin{pmatrix} 5 & 6 \\ 6 & \cancel{6}\ 10 \end{pmatrix} \begin{pmatrix} \cancel{4}\ 2 \\ -1 \end{pmatrix}$

~~$= \frac{1}{14}\begin{pmatrix} 14 \\ 24 \\ 14 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$~~

$= \frac{1}{14} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

~~$= \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$~~

4. [5pt] Let

$$\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Apply the Gram–Schmidt algorithm to the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and obtain an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$\operatorname{proj}_y(x) = \frac{\langle x, y\rangle}{|y|^2}\vec{y}$$

G-S algorithm

$$\vec{u}_1 = \vec{v}_1 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \operatorname{proj}_{u_1}(\vec{v}_2) = \vec{v}_2 - \frac{1}{2}\vec{u}_1$$

$$= \begin{pmatrix}\cancel{0}1\\1\\0\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1/2\\-1/2\\0\end{pmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \operatorname{proj}_{\vec{u}_1}(\vec{v}_3) - \operatorname{proj}_{\vec{u}_2}(\vec{v}_3)$$

$$= \vec{v}_3 - 0\cdot\vec{u}_1 - \frac{1}{3/2}\cdot\vec{u}_2$$

$$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1/2\\-1/2\\0\end{pmatrix} = \cancel{\begin{pmatrix}1/2\\-1/4\\1/4\\0\end{pmatrix}} = \begin{pmatrix}1/3\\-1/3\\1/3\\0\end{pmatrix}$$

5. [5pt] Let

$$A = \begin{bmatrix} x & 4 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

where x is a real number. Find $x \in \mathbb{R}$ such that $\det(A) = 0$.

Expand the first row.

$$\det(A) = x \cdot \det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - 4 \cdot \det\begin{pmatrix} 4 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= 2x - 4 \cdot (-4)$$

$$= 2x + 16$$

If $2x+16=0$, then $\underline{\underline{x=-8}}$.

6. [5pt] Let

$$A = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}.$$

Show that $\det(A) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$. Make sure to justify every step of your argument.

$$\det(A) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix}$$

(每列扣掉第一列)

$$= \begin{vmatrix} b-a & b^2-a^2 & b^3-a^3 \\ c-a & c^2-a^2 & c^3-a^3 \\ d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} = (b-a)(c-a)(d-a) \begin{vmatrix} 1 & b+a & b^2+ab+a^2 \\ 1 & c+a & c^2+ac+a^2 \\ 1 & d+a & d^2+ad+a^2 \end{vmatrix}$$

(每列提出公因式)

$$= (b-a)(c-a)(d-a) \times \begin{vmatrix} 1 & b+a & b^2+ab+a^2 \\ 0 & c-b & (c-b)(c+b+a) \\ 0 & d-b & (d-b)(d+b+a) \end{vmatrix}$$

(每列扣第一列)

$$= \cancel{\begin{vmatrix} 1 & & \\ 0 & 1 & c+b+a \\ 0 & 1 & d+b+a \end{vmatrix}} = (b-a)(c-a)(d-a) \times \begin{vmatrix} c-b & (c-b)(c+b+a) \\ d-b & (d-b)(d+b+a) \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b) \times \begin{vmatrix} 1 & c+b+a \\ 1 & d+b+a \end{vmatrix}$$

(每列提出公因式)

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

7. [5pt] Let J_n be the $n \times n$ all-ones matrix. Let I_n be the $n \times n$ identity matrix. Find $\det(J_n - I_n)$ as a formula of n.

$$|J_n - I_n| = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ n-1 & 0 & & & \\ \vdots & & 0 & & 1 \\ \vdots & & 1 & \ddots & \\ n-1 & & & & 0 \end{vmatrix}$$

(每行加到第一行)

$$= (n-1) \begin{vmatrix} 1 & 0 & & 1 \\ \vdots & & \ddots & \\ \vdots & 1 & & \ddots \\ 1 & & & 0 \end{vmatrix}$$

第一行
(提出 $n-1$)

$$= (n-1) \begin{vmatrix} 1 & -1 & & 0 \\ \vdots & & -1 & \\ \vdots & 0 & & \ddots \\ 1 & & & -1 \end{vmatrix}$$

(用第一行去扣每一行)

$$= (n-1)(-1)^{n-1}.$$

8. [extra 2pt] Let

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccccccccc} \uparrow & \uparrow & & & - - - & & & & \uparrow \\ v_1 & v_2 & & & & & & & \vec{v}_9 \end{array} \end{array}.$$

Find $\det(A)$.

$\det(A) = 0$ since $\vec{v}_1 + \vec{v}_4 + \vec{v}_7 = \begin{pmatrix} \vdots \end{pmatrix} = \vec{v}_2 + \vec{v}_5 + \vec{v}_8$.

and $\vec{v}_1 + \vec{v}_4 + \vec{v}_7 - \vec{v}_2 + \vec{v}_5 + \vec{v}_8 = \vec{0}$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	2	
Total	30 (+2)	