Sample Solutions for Sample Questions 10.

1. Compute $A\vec{v}_1 = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\vec{v}_1$

$A\vec{v}_2 = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0\,\vec{v}_2$

$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 0.$

Thus, $A\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \begin{pmatrix}2&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

$\Rightarrow Q = \begin{pmatrix}1&1\\1&-1\end{pmatrix}$ is invertible

$D = \begin{pmatrix}2&0\\0&0\end{pmatrix}$ is diagonal.

2. $A - xI = \begin{pmatrix}1-x&1\\1&1-x\end{pmatrix} \Rightarrow p(x) = \det(A - xI) = (1-x)^2 - 1$

$= x^2 - 2x = x(x-2).$

If $p(x) = 0$, then $x = 0, 2$.

3. "$\lambda = 0$" : ~~Solve A~~ $A - \lambda I = \begin{pmatrix}1&1\\1&1\end{pmatrix}$

Solve $\begin{pmatrix}1&1\\1&1\end{pmatrix}\vec{v} = \vec{0}$ $\quad \begin{pmatrix}1&1\\1&1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1\\0&0\end{pmatrix}$

$\uparrow$ free.

$\Rightarrow \vec{v} = \begin{pmatrix}-1\\1\end{pmatrix}$

"$\lambda = 2$" $A - \lambda I = \begin{pmatrix}-1&1\\1&-1\end{pmatrix}$

Solve $\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\vec{v} = \vec{0}$ $\quad \begin{pmatrix}-1&1\\1&-1\end{pmatrix} \rightsquigarrow \begin{pmatrix}-1&1\\0&0\end{pmatrix}$

$\uparrow$ free

$\Rightarrow \vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$

4. $A = \begin{pmatrix} -2 & 15 \\ 1 & 0 \end{pmatrix}$.

① Find eigenvalues.

$$\det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & 15 \\ 1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 + 2\lambda - 15 = (\lambda - 3)(\lambda + 5).$$

$\Rightarrow \lambda = 3, -5.$

② Find eigenvectors for each eigenvalue.

(i) $\lambda = 3$. $A - 3I = \begin{pmatrix} -5 & 15 \\ 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.

(ii). $\lambda = -5$, $A + 5I = \begin{pmatrix} 3 & 15 \\ 1 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 5 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} 5 \\ -1 \end{pmatrix} \right\}$.

Thus, $AQ = QD$ with $Q = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & \\ & -5 \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$.

5. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

① Find eigenvalues.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

$$= (\lambda + i)(\lambda - i).$$

$\Rightarrow \lambda = \pm i.$

[這個A無法在ℝ中對角化].

② Find eigenvector for each eigenvalue.

$\lambda = i.$ $A - iI = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$

$\Rightarrow$ ~~nullspace~~ nullspace $= \text{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$

$\lambda = -i.$ $A + iI = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}.$

Thus, $AQ = QD$ with $Q = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} i & \\ & -i \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D.$

6.

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \\ -1 & & 1 \end{pmatrix}$$

① Find eigenvalues

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 1-\lambda & \\ -1 & & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2 - 3\lambda + 2 \quad -2\right]$$

$$= (1-\lambda)(\lambda-3)\cdot\lambda \quad \Rightarrow \lambda = 0, 1, 3.$$

② Find eigenvectors for each eigenvalue.

(i) $\lambda = 0$. $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \\ -1 & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & \\ & -1 & 1 \\ & & \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

(ii) $\lambda = 1$ $A - I = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & \\ -1 & & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ nullspace $= \text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$.

(iii) $\lambda = 3$, $A - 3I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2 & \\ -1 & & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ & 1 & -1 \\ & & \end{pmatrix}$

$\Rightarrow$ null space $= \text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$.

Thus, $AQ = QD$ with $Q = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 3 \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$.

7. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

① Find eigenvalues.

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} =$$

$$= -(\lambda+1)^2 \cdot (\lambda-2). \quad \Rightarrow \lambda = -1, -1, 2.$$

② Find eigenvector for each eigenvalue.

(i) $\lambda = -1$. $A + I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$\Rightarrow$ nullspace $= \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\lambda=-1$ 有 2個重根, eigenvector "通常" 有兩個.

(ii) $\lambda = 2$. $A - 2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & -1 \\ & 1 & -1 \\ & & \end{pmatrix}$

$\Rightarrow$ ~~null~~ nullspace $= \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Thus, $AQ = QD$ with $Q = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

$\Updownarrow$

$Q^{-1}AQ = D$