Sample Solutions for Sample Questions 12

1. ~~Check~~

Since t is the orthogonal projection onto V,

$$t(\vec{v}_1) = \vec{v}_1$$
$$t(\vec{v}_2) = \vec{v}_2$$
$$t(\vec{v}_3) = 0 \cdot \vec{v}_3$$

Let $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Then $\mathrm{Rep}_{B,B}(t) = \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}}_{T}$.

$\Rightarrow$ eigenvalues $= 1, 1, 0$.

eigenspaces :

$$E_1 = \{\vec{v} \mid t(\vec{v}) = 1 \cdot \vec{v}\}$$
$$= \mathrm{span}\{\vec{v}_1, \vec{v}_2\}.$$
$$E_0 = \{\vec{v} \mid t(\vec{v}) = 0 \cdot \vec{v}\}.$$
$$= \mathrm{span}\{\vec{v}_3\}.$$

characteristic polynomial

$$p(x) = \det(T - xI) = (1-x)(1-x)(0-x).$$

2.

Reflection: $\begin{cases} t(\vec{v}_1) = \vec{v}_1 \\ t(\vec{v}_2) = \vec{v}_2 \\ t(\vec{v}_3) = -\vec{v}_3 \end{cases}$

Let $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be an orthonormal basis of $\mathbb{R}^3$.

$$\Rightarrow \quad \mathrm{Rep}_{B,B}(t) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

3. [這題跟筆記一樣,練習寫一下]

Suppose A and B are similar.

$\Rightarrow \exists\, Q$ (Q invertible) such that $Q^{-1}AQ = B$.

Since $\det(Q^{-1}) \cdot \det(Q) = \det(I) = 1$,

char poly of A $= \det(A - xI)$

$= \det(Q^{-1}) \det(A - xI) \cdot \det(Q)$

$= \det(Q^{-1}(A - xI)Q)$

$= \det(Q^{-1}AQ - xQ^{-1}IQ)$

$= \det(B - xI)$ = char poly of ~~A~~ B.

4. [This example show you that you do not always have to use the char poly to find the eigenvalues.].

① Observe that

$$J_n \cdot 1_n = n \cdot 1_n$$

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix}.$$

$\Rightarrow$ n is an eigenvalue of geo mult at least 1.

② Observe that rank $J_n = 1$.

$\Rightarrow$ dim nullspace $(J_n) = n-1$

$\Rightarrow$ dim $E_0 = n-1$.

eigenspace with respect to 0.

Now

eigenvalues	0	n
geo mult	$n-1$	≥ 1
alg mult	$\geq n-1$	≥ 1

But $p(x) = \det(J_n - xI)$ is a poly of degree n.

$$\Rightarrow p(x) = (-1)^n (x-0)^{n-1} \cdot (x-n)^1 = (-1)^n \cdot x^{n-1}(x-n)$$

the leading coefficient of $\det(A - xI)$ for an $n\times n$ matrix A is always $(-1)^n$.

5.

① Compute an eigenvalue.

$$\det(A - xI) = \det\begin{pmatrix} -x & 1 & 1 \\ -1 & 2-x & 2 \\ 0 & 0 & 1-x \end{pmatrix} = (x^2 - 2x + 1)(1-x) = -(x-1)^3.$$

Let $\lambda = 1$. Find an eigenvector.

$$A - I = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

free.

$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$. Pick a unit eigenvector $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Expand $\{\vec{v}_1\}$ to an orthonormal basis

$$\left\{\vec{v}_1, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}\right\}$$

可以用 Gram-Schmidt 或用湊的.

$$Q^{(1)} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{(1)} = [Q^{(1)}]^* \, T \, Q^{(1)} = \begin{pmatrix} 1 & -2 & 3/\sqrt{2} \\ 0 & 1 & -1/\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

(Here T = A.)

因為 $T\vec{v}_1 = \vec{v}_1$

$$T \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \underbrace{-\frac{2}{\sqrt{2}}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \underbrace{\frac{1}{\sqrt{2}}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \underbrace{\frac{3}{\sqrt{2}}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \underbrace{-\frac{1}{\sqrt{2}}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \underbrace{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

也可以𢖫直接計算

[5 continued]

Let $Q = Q^{(1)} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $Q^T A Q$ is upper-triangular.

6. Suppose $U = [u_{i,j}]$.

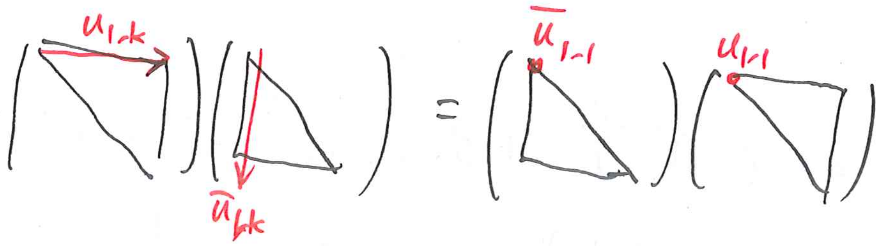
U矩陣的 i,j-entry [illegible]叫 $u_{i,j}$.

Since U is upper triangular,

$$u_{i,j} = 0 \text{ if } i > j. \quad \cdots\cdots (1).$$

Compare the 1,1-entry of UU^* and U^*U.

$$\Rightarrow \sum_{k=1}^{n} u_{1,k} \bar{u}_{1,k} = \bar{u}_{1,1} \cdot u_{1,1}$$



$$\Rightarrow |u_{1,1}|^2 + |u_{1,2}|^2 + \cdots + |u_{1,k}|^2 = |u_{1,1}|^2$$

$$\Rightarrow u_{1,2} = u_{1,3} = \cdots = u_{1,k} = 0.$$

Next compare the 2,2-entry, the 3,3-entry, ...

$\Rightarrow U$ is diagonal.

7. Do diagonalization on your own.

$$\lambda = 0 \Rightarrow E_0 = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}. \quad \text{Pick } \vec{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\lambda = 1 \Rightarrow E_1 = \text{span}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}. \quad \text{Pick } \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\lambda = 3 \Rightarrow E_3 = \text{span}\left\{\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\}. \quad \text{Pick } \vec{v}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

$$\Rightarrow \text{Let } Q = \begin{pmatrix}| & | & |\\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3\\ | & | & |\end{pmatrix} \quad D = \begin{pmatrix}0 & & \\ & 1 & \\ & & 3\end{pmatrix}$$

$$\Rightarrow Q^T A Q = D.$$

$$\Rightarrow A = Q D Q^T$$

$$= \begin{pmatrix}| & | & |\\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3\\ | & | & |\end{pmatrix}\begin{pmatrix}0 & & \\ & 1 & \\ & & 3\end{pmatrix}\begin{pmatrix}- & \vec{v}_1 & -\\ - & \vec{v}_2 & -\\ - & \vec{v}_3 & -\end{pmatrix}$$

$$= 0 \cdot \vec{v}_1 \vec{v}_1^T + 1 \cdot \vec{v}_2 \vec{v}_2^T + 3 \cdot \vec{v}_3 \vec{v}_3^T$$

$$= 0 \cdot \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} + 1 \cdot \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{pmatrix} + 3 \cdot \frac{1}{6}\begin{pmatrix}4 & -2 & -2\\ -2 & 1 & 1\\ -2 & 1 & 1\end{pmatrix}$$